Chapter 13: Grammatical Format

I. Theory of Automata

→ II. Theory of Formal Languages

III. Theory of Turing Machines …

Theorem. All regular languages are context-free languages.

Proof. We show that for any FA, there is a CFG such that the language generated by the grammar is the same as the language accepted by the FA.

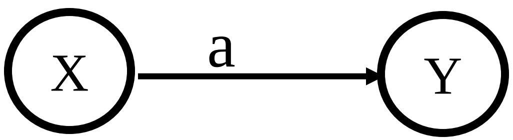
By constructive algorithm.

Input: a finite automaton.

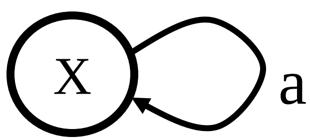
Output: a CF grammar.

- The alphabet Σ of terminals is the alphabet of the FA.
- Nonterminals are the state names. (The start state is renamed S.)

- For every edge, create a production:



$X \rightarrow aY$

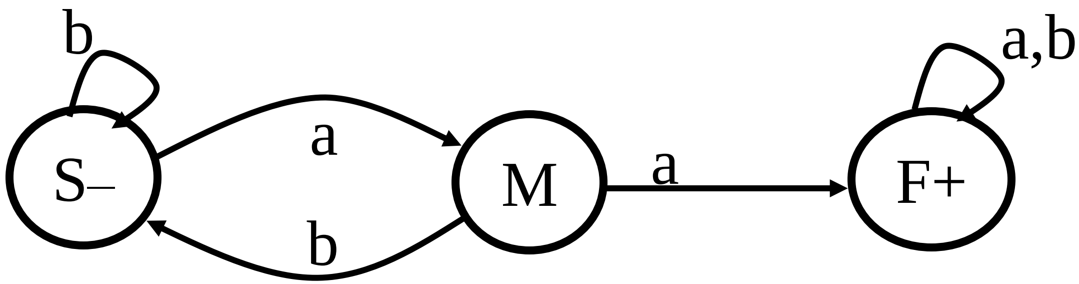


$X \rightarrow aX$

- For every final state, create a production:

$X \rightarrow \Lambda$

Example:



$S \to aM \,/\, bS$

$M \to aF /\, bS$

$F \to aF /\, bF /\, \Lambda$

babbaaba

$$S \xrightarrow{b} S \xrightarrow{a} M \xrightarrow{b} S \xrightarrow{b} S \xrightarrow{a} M \xrightarrow{a} F \xrightarrow{b} F \xrightarrow{a} F$$

$$S \Rightarrow bS \Rightarrow baM \Rightarrow babS \Rightarrow babbS \Rightarrow babbaM \Rightarrow babbaaF$$
$$\Rightarrow babbaabF \Rightarrow babbaabaF \Rightarrow babbaaba$$

Definition: A semiword is a sequence of terminals (possibly none) followed by exactly one nonterminal.

(terminal)(terminal)…(terminal)(Nonterminal)

Definition: A CFG is a regular grammar if all of its productions have the form:

Nonterminal $\rightarrow$ semiword

or

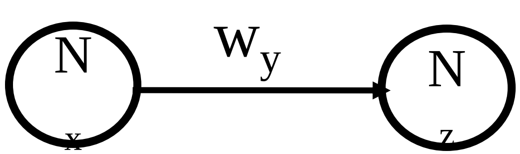
Nonterminal $\rightarrow$ word (a sequence of terminals or Λ)

Theorem. All languages generated by regular grammars are regular.

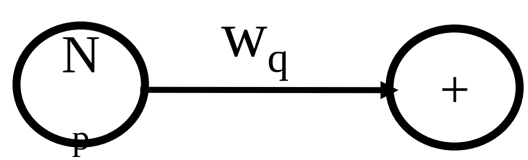
Proof. By constructive algorithm. We build a transition graph.

- The alphabet ∑ of the transition graph is the set of terminals.
- One state for each nonterminal. The state named S is the start state. We add one final state +.
- Transitions:

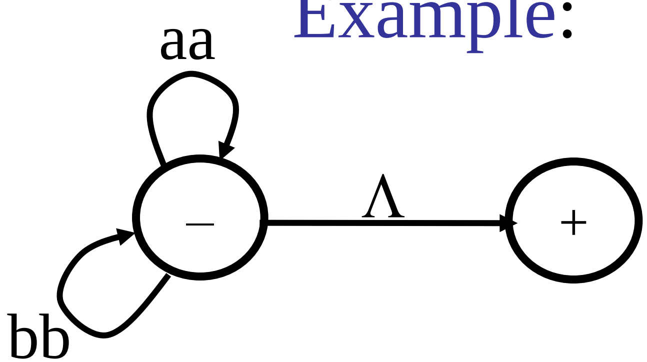
$N_x \rightarrow w_y N_z$



$N_p \rightarrow w_q$

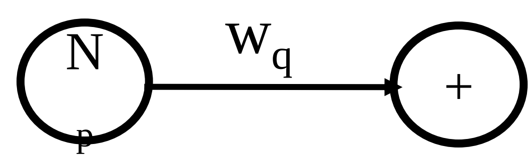


Example: $S \to aaS \quad S \to bbS \quad S \to \Lambda$



(aa+bb)*

Alternative Algorithm:

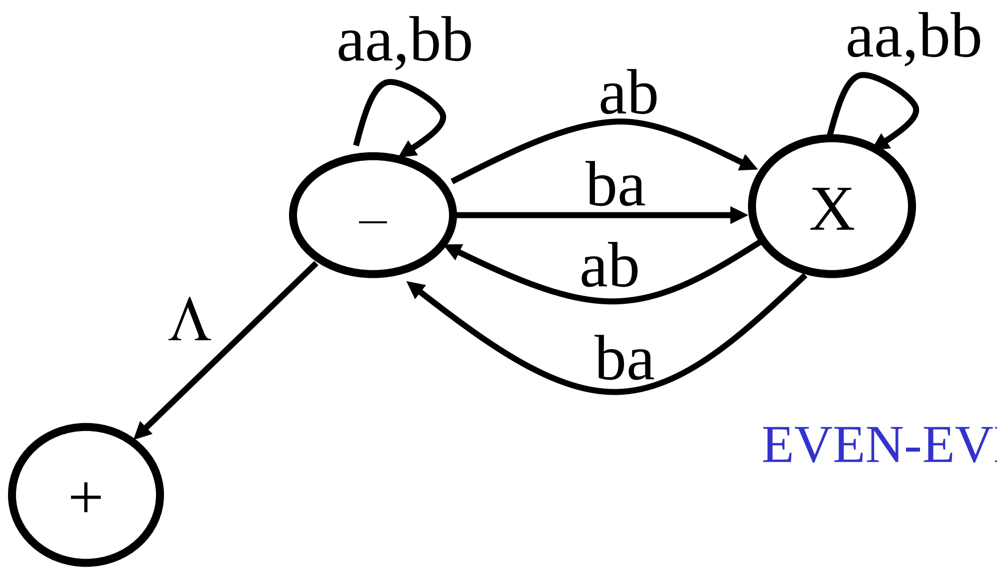


1. $N_p \to w_q$ whenever w_q is not Λ
2. For each transition of the form $N \to \Lambda$, we mark the state for N with +..

Example

$S \to aaS \ / \ bbS \ / \ abX \ / \ baX \ / \ \Lambda$

$X \to aaX \ / \ bbX \ / \ abS \ / \ baS$



EVEN-EVEN

- If the empty word is in the language, a production of the form $N \to \Lambda$ (called a Λ-production) is necessary.
- The existence of a production of the form $N \to \Lambda$ des not necessarily mean that Λ is a part of the language.

$$S \to aX \quad S \to a \quad X \to \Lambda$$

Theorem. Let L be a language generated by a CFG. There exists a CFG without productions of the form $X \to \Lambda$ such that:

1. If $\Lambda \notin L$, L is generated by the new grammar.
2. If $\Lambda \in L$, all words of L except for Λ are generated by the new grammar.

Definition. A production of the form:

Nonterminal → one Nonterminal is called a unit production.

Theorem. Let L be a language generated by a CFG that has no Λ-productions. Then there is another CFG without Λ -productions and without unit productions that generates L.

Theorem. Let L be a language generated by a CFG. The there exists another CFG that generates all words of L (except Λ) such that all the productions are of the form:

Nonterminal → sequence of Nonterminals

Nonterminal → one terminal

<u>Definition.</u> A CFG in is said to be in Chomsky Normal Form (CNF) if all the productions have the form:

Nonterminal → (Nonterminal)(Nonterminal)

Nonterminal → terminal

<u>Theorem.</u> Let L be a language generated by a CFG. There there is another grammar which is in CNF that generates all the words of L (except Λ).

Definition. In a sequence of terminals and nonterminals in a derivation (called a working string), if there is at least one nonterminal, then the first one is called the leftmost nonterminal.

Definition. A leftmost derivation is a derivation where at each step, a production is applied to the leftmost nonterminal in the working string.

Example. $S \rightarrow aSX \mid b$

$X \rightarrow Xb \mid a$

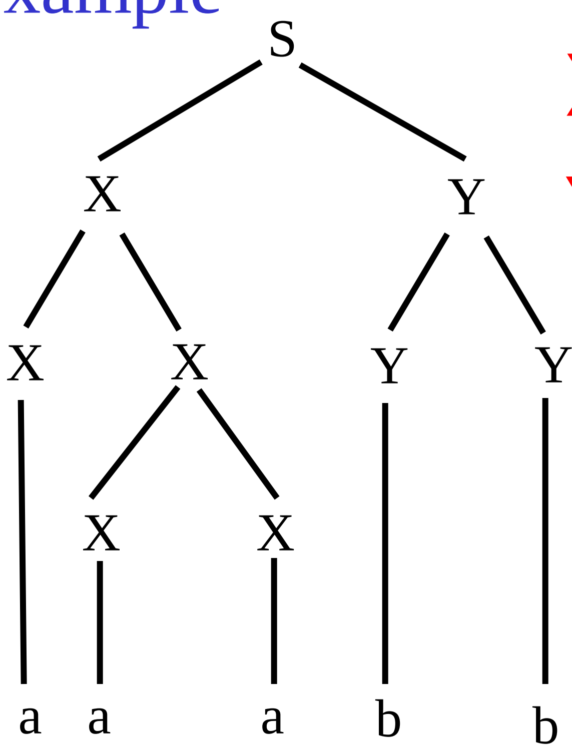
$S \Rightarrow aSX \Rightarrow aaSXX \Rightarrow aabXX \Rightarrow aabXbX \Rightarrow aababX \Rightarrow aababa$

Example

$S \rightarrow XY$

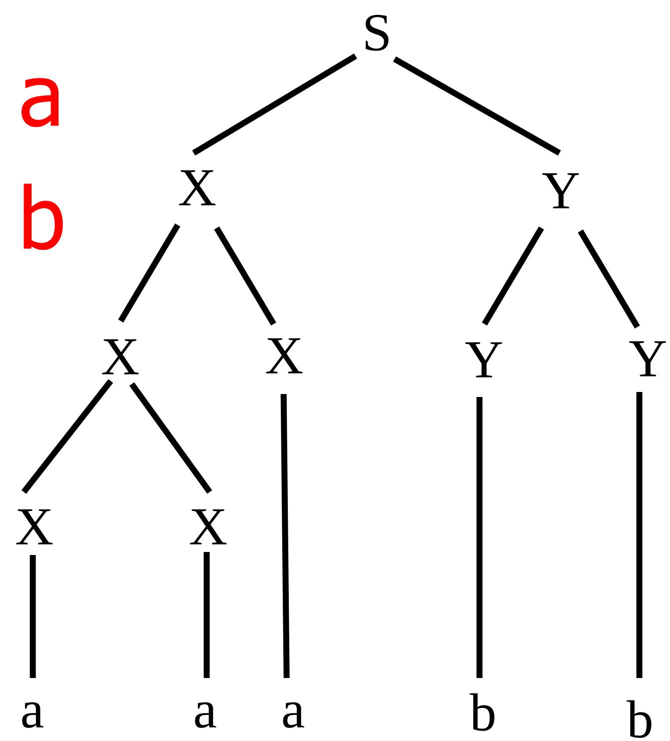
$X \rightarrow XX \mid a$

$Y \rightarrow YY \mid b$



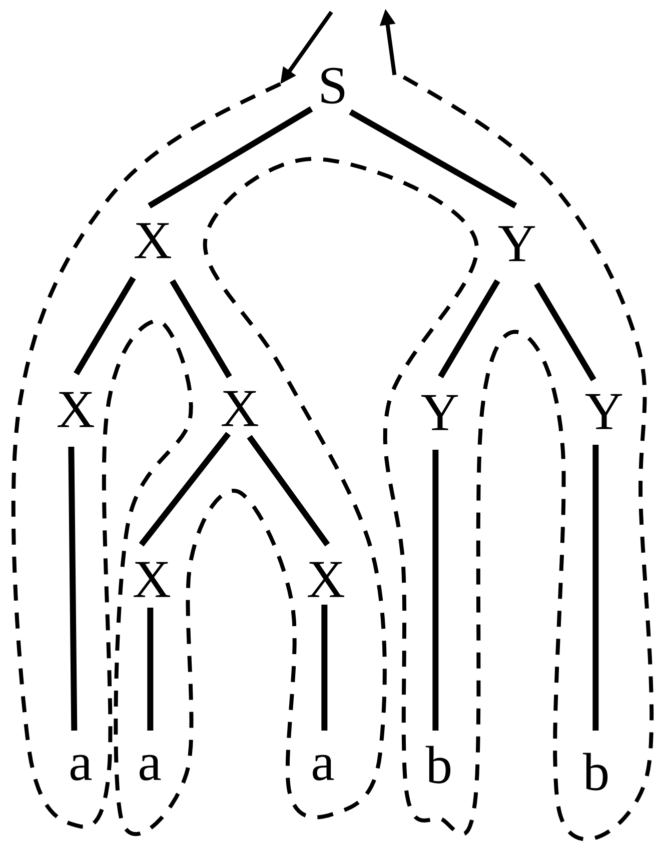
Derivation I

aabb

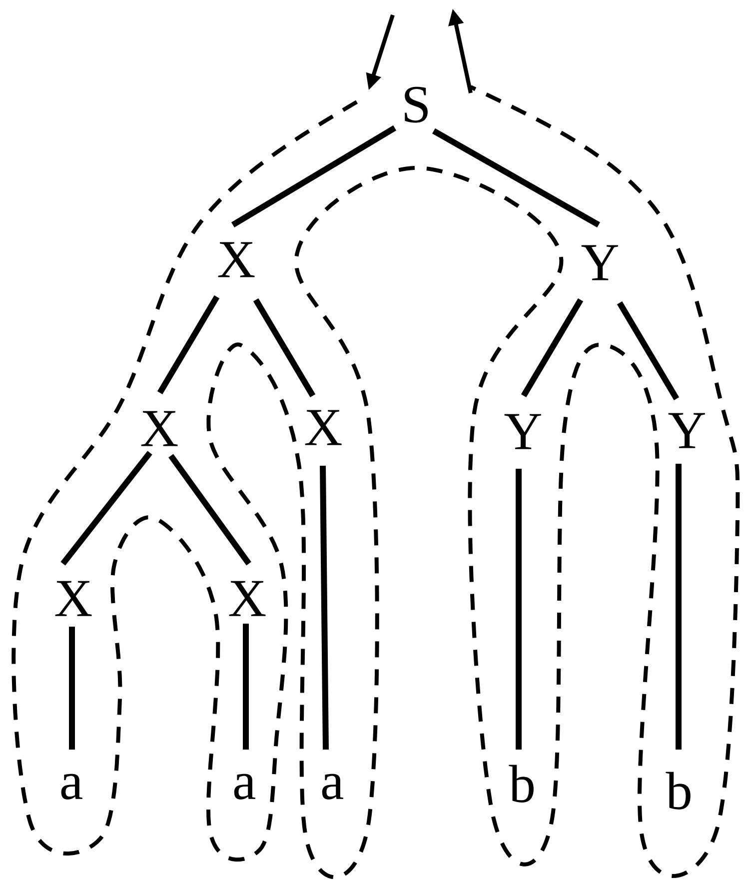


Derivation II

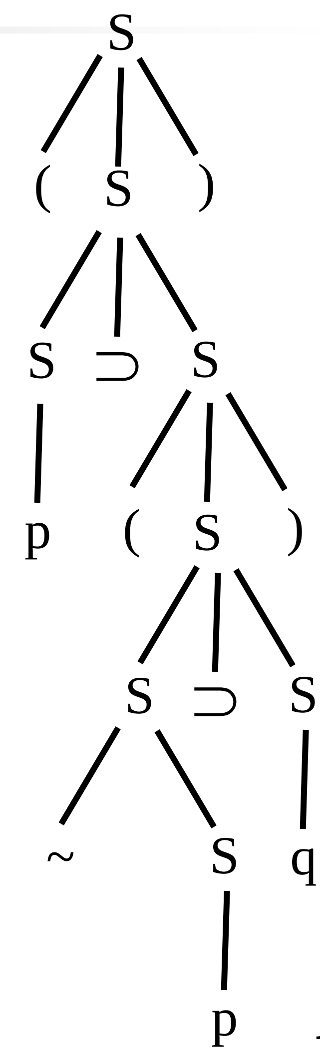
$$
\begin{aligned}
S &\Rightarrow XY \\
&\Rightarrow XXY \\
&\Rightarrow aXY \\
&\Rightarrow aXXY \\
&\Rightarrow aaXY \\
&\Rightarrow aaaY \\
&\Rightarrow aaaYY \\
&\Rightarrow aaabY \\
&\Rightarrow aaabb
\end{aligned}
$$



$$
\begin{aligned}
S &\Rightarrow XY \\
&\Rightarrow XXY \\
&\Rightarrow XXXY \\
&\Rightarrow aXXY \\
&\Rightarrow aaXY \\
&\Rightarrow aaaY \\
&\Rightarrow aaaYY \\
&\Rightarrow aaabY \\
&\Rightarrow aaabb
\end{aligned}
$$



Theorem. Any word that is in the language generated by a CFG has a leftmost derivation.



Example. $S \rightarrow (S) \mid S \supset S \mid {\sim}S \mid p \mid q$

$$
\begin{aligned}
S \; & \Rightarrow (S) \Rightarrow (S \supset S) \Rightarrow (p \supset S) \\
& \Rightarrow (p \supset (S)) \Rightarrow (p \supset (S \supset S)) \\
& \Rightarrow (p \supset ({\sim}S \supset S)) \\
& \Rightarrow (p \supset ({\sim}p \supset S)) \\
& \Rightarrow (p \supset ({\sim}p \supset q))
\end{aligned}
$$