Solution to Chapter 13 Problems

Problem 13.1

The codewords of the linear code of Example 13.2.1 are

$$\begin{aligned}\mathbf{c}_1 &= [\,0\ \ 0\ \ 0\ \ 0\ \ 0\,]\\ \mathbf{c}_2 &= [\,1\ \ 0\ \ 1\ \ 0\ \ 0\,]\\ \mathbf{c}_3 &= [\,0\ \ 1\ \ 1\ \ 1\ \ 1\,]\\ \mathbf{c}_4 &= [\,1\ \ 1\ \ 0\ \ 1\ \ 1\,]\end{aligned}$$

Since the code is linear the minimum distance of the code is equal to the minimum weight of the codewords. Thus,

$$d_{\min} = w_{\min} = 2$$

There is only one codeword with weight equal to 2 and this is $\mathbf{c}_2$.

Problem 13.2

The parity check matrix of the code in Example 13.2.3 is

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The codewords of the code are

$$\begin{aligned}\mathbf{c}_1 &= [\,0\ \ 0\ \ 0\ \ 0\ \ 0\,]\\ \mathbf{c}_2 &= [\,1\ \ 0\ \ 1\ \ 0\ \ 0\,]\\ \mathbf{c}_3 &= [\,0\ \ 1\ \ 1\ \ 1\ \ 1\,]\\ \mathbf{c}_4 &= [\,1\ \ 1\ \ 0\ \ 1\ \ 1\,]\end{aligned}$$

Any of the previous codewords when postmultiplied by $\mathbf{H}^t$ produces an all-zero vector of length 3. For example

$$\begin{aligned}\mathbf{c}_2\mathbf{H}^t &= [\,1\oplus 1\ \ 0\ \ 0\,] = [\,0\ \ 0\ \ 0\,]\\ \mathbf{c}_4\mathbf{H}^t &= [\,1\oplus 1\ \ 1\oplus 1\ \ 1\oplus 1\,] = [\,0\ \ 0\ \ 0\,]\end{aligned}$$

Problem 13.3

The following table lists all the codewords of the (7,4) Hamming code along with their weight. Since the Hamming codes are linear $d_{\min} = w_{\min}$. As it is observed from the table the minimum weight is 3 and therefore $d_{\min} = 3$.

No.	Codewords	Weight
1	0000000	0
2	1000110	3
3	0100011	3
4	0010101	3
5	0001111	4
6	1100101	4
7	1010011	4
8	1001001	3
9	0110110	4
10	0101100	3
11	0011010	3
12	1110000	3
13	1101010	4
14	1011100	4
15	0111001	4
16	1111111	7

Problem 13.4

The parity check matrix **H** of the (15,11) Hamming code consists of all binary sequences of length 4, except the all zero sequence. The systematic form of the matrix **H** is

$$\mathbf{H} = [\ \mathbf{P}' \ \mid \ \mathbf{I}_4\ ] = \left(\begin{array}{ccccccccccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

The corresponding generator matrix is

$$\mathbf{G} = [\ \mathbf{I}_{11}\ \mid\ \mathbf{P}\ ] = \left(\begin{array}{ccccccccccc|cccc} 1 & & & & & & & & & & & 1 & 1 & 0 & 0 \\ & 1 & & & & & & & & & & 1 & 0 & 1 & 0 \\ & & 1 & & & & & \mathbf{0} & & & & 1 & 0 & 0 & 1 \\ & & & 1 & & & & & & & & 0 & 1 & 1 & 0 \\ & & & & 1 & & & & & & & 0 & 1 & 0 & 1 \\ & & & & & 1 & & & & & & 0 & 0 & 1 & 1 \\ & & & & & & 1 & & & & & 1 & 1 & 1 & 0 \\ & & & & & & & 1 & & & & 1 & 1 & 0 & 1 \\ & & & \mathbf{0} & & & & & 1 & & & 1 & 0 & 1 & 1 \\ & & & & & & & & & 1 & & 0 & 1 & 1 & 1 \\ & & & & & & & & & & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

Problem 13.5

Let C be an (n, k) linear block code with parity check matrix $\mathbf{H}$. We can express the parity check matrix in the form

$$\mathbf{H} = [\ \mathbf{h}_1\ \ \mathbf{h}_2\ \ \cdots\ \ \mathbf{h}_n\ ]$$

where $\mathbf{h}_i$ is an $n - k$ dimensional column vector. Let $\mathbf{c} = [c_1 \cdots c_n]$ be a codeword of the code C with l nonzero elements which we denote as $c_{i_1}, c_{i_2}, \ldots, c_{i_l}$. Clearly $c_{i_1} = c_{i_2} = \ldots = c_{i_l} = 1$ and since $\mathbf{c}$ is a codeword

$$\begin{aligned} \mathbf{c}\mathbf{H}^t = 0 &= c_1\mathbf{h}_1 + c_2\mathbf{h}_2 + \cdots + c_n\mathbf{h}_n \\ &= c_{i_1}\mathbf{h}_{i_1} + c_{i_2}\mathbf{h}_{i_2} + \cdots + c_{i_l}\mathbf{h}_{i_l} \\ &= \mathbf{h}_{i_1} + \mathbf{h}_{i_2} + \cdots + \mathbf{h}_{i_l} = 0 \end{aligned}$$

This proves that l column vectors of the matrix $\mathbf{H}$ are linear dependent. Since for a linear code the minimum value of l is $w_{\min}$ and $w_{\min} = d_{\min}$, we conclude that there exist $d_{\min}$ linear dependent column vectors of the matrix $\mathbf{H}$.

Now we assume that the minimum number of column vectors of the matrix $\mathbf{H}$ that are linear dependent is $d_{\min}$ and we will prove that the minimum weight of the code is $d_{\min}$. Let $\mathbf{h}_{i_1}, \mathbf{h}_{i_2}, \ldots, \mathbf{h}_{d_{\min}}$ be a set of linear dependent column vectors. If we form a vector $\mathbf{c}$ with non-zero components at positions $i_1, i_2, \ldots, i_{d_{\min}}$, then

$$\mathbf{c}\mathbf{H}^t = c_{i_1}\mathbf{h}_{i_1} + \cdots + c_{i_{d_{\min}}} = 0$$

which implies that $\mathbf{c}$ is a codeword with weight $d_{\min}$. Therefore, the minimum distance of a code is equal to the minimum number of columns of its parity check matrix that are linear dependent.

For a Hamming code the columns of the matrix $\mathbf{H}$ are non-zero and distinct. Thus, no two columns $\mathbf{h}_i$, $\mathbf{h}_j$ add to zero and since $\mathbf{H}$ consists of all the $n - k$ tuples as its columns, the sum $\mathbf{h}_i + \mathbf{h}_j = \mathbf{h}_m$ should also be a column of $\mathbf{H}$. Then,

$$\mathbf{h}_i + \mathbf{h}_j + \mathbf{h}_m = 0$$

and therefore the minimum distance of the Hamming code is 3.

Problem 13.6

The generator matrix of the $(n, 1)$ repetition code is a $1 \times n$ matrix, consisted of the non-zero codeword. Thus,

$$\mathbf{G} = \left[\begin{array}{c|cccc} 1 & 1 & \cdots & 1 \end{array} \right]$$

This generator matrix is already in systematic form, so that the parity check matrix is given by

$$\mathbf{H} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{array} \right)$$

Problem 13.7

1) The parity check matrix $\mathbf{H}_e$ of the extended code is an $(n+1-k) \times (n+1)$ matrix. The codewords of the extended code have the form

$$\mathbf{c}_{e,i} = [\ \mathbf{c}_i \ \mid \ x\ ]$$

where x is 0 if the weight of $\mathbf{c}_i$ is even and 1 if the weight of $\mathbf{c}_i$ is odd. Since $\mathbf{c}_{e,i}\mathbf{H}_e^t = [\mathbf{c}_i|x]\mathbf{H}_e^t = 0$ and $\mathbf{c}_i\mathbf{H}^t = 0$, the first $n-k$ columns of $\mathbf{H}_e^t$ can be selected as the columns of $\mathbf{H}^t$ with a zero added in the last row. In this way the choice of x is immaterial. The last column of $\mathbf{H}_e^t$ is selected in such a way that the even-parity condition is satisfied for every codeword $\mathbf{c}_{e,i}$. Note that if $\mathbf{c}_{e,i}$ has even weight, then

$$c_{e,i_1} + c_{e,i_2} + \cdots + c_{e,i_{n+1}} = 0 \Longrightarrow \mathbf{c}_{e,i}[\ 1 \ \ 1 \ \ \cdots \ \ 1\ ]^t = 0$$

for every i. Therefore the last column of $\mathbf{H}_e^t$ is the all-one vector and the parity check matrix of the extended code has the form

$$\mathbf{H}_e = (\mathbf{H}_e^t)^t = \left(\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right)^t = \left(\begin{array}{ccccccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right)$$

2) The original code has minimum distance equal to 3. But for those codewords with weight equal to the minimum distance, a 1 is appended at the end of the codewords to produce even parity. Thus, the

In this case coding has decreased the error probability by a factor of 6.

Problem 13.9

The following table shows the standard array for the (7,4) Hamming code.

		$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
		1000000	0100000	0010000	0001000	0000100	0000010	0000001
$\mathbf{c}_1$	0000000	1000000	0100000	0010000	0001000	0000100	0000010	0000001
$\mathbf{c}_2$	1000110	0000110	1100110	1010110	1001110	1000010	1000100	1000111
$\mathbf{c}_3$	0100011	1100011	0000011	0110011	0101011	0100111	0100001	0100010
$\mathbf{c}_4$	0010101	1010101	0110101	0000101	0011101	0010001	0010111	0010100
$\mathbf{c}_5$	0001111	1001111	0101111	0011111	0000111	0001011	0001101	0001110
$\mathbf{c}_6$	1100101	0100101	1000101	1110101	1101101	1100001	1100111	1100100
$\mathbf{c}_7$	1010011	0010011	1110011	1000011	1011011	1010111	1010001	1010010
$\mathbf{c}_8$	1001001	0001001	1101001	1011001	1000001	1001101	1001011	1001000
$\mathbf{c}_9$	0110110	1110110	0010110	0100110	0111110	0110010	0110100	0110111
$\mathbf{c}_{10}$	0101100	1101100	0001100	0111100	0100100	0101000	0101110	0101101
$\mathbf{c}_{11}$	0011010	1011010	0111010	0001010	0010010	0011110	0011000	0011011
$\mathbf{c}_{12}$	1110000	0110000	1010000	1100000	1111000	1110100	1110010	1110001
$\mathbf{c}_{13}$	1101010	0101010	1001010	1111010	1100010	1101110	1101000	1101011
$\mathbf{c}_{14}$	1011100	0011100	1111100	1001100	1010100	1011000	1011110	1011101
$\mathbf{c}_{15}$	0111001	1111001	0011001	0101001	0110001	0111101	0111011	0111000
$\mathbf{c}_{16}$	1111111	0111111	1011111	1101111	1110111	1111011	1111101	1111110

As it is observed the received vector $\mathbf{y} = [1110100]$ is in the 7^{th} column of the table under the error vector $\mathbf{e}_5$. Thus, the received vector will be decoded as

$$\mathbf{c} = \mathbf{y} + \mathbf{e}_5 = [\ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ ] = \mathbf{c}_{12}$$

Problem 13.10

1) Since for each time slot $[mT, (m+1)T]$ we have $\psi_1(t) = \pm\psi_2(t)$, the signals are dependent and thus only one dimension is needed to represent them in the interval $[mT, (m+1)T]$. In this case the dimensionality of the signal space is upper bounded by the number of the different time slots used to transmit the message signals.

2) If $\psi_1(t) \neq \alpha\psi_2(t)$, then the dimensionality of the signal space over each time slot is at most 2. Since there are n slots over which we transmit the message signals, the dimensionality of the signal space is upper bounded by $2n$.

3) Let the decoding rule be that the first codeword is decoded when $\mathbf{r}$ is received if

$$p(\mathbf{r}|\mathbf{x}_1) > p(\mathbf{r}|\mathbf{x}_2)$$