Solution to Chapter 8 Problems

Problem 8.1

1) To show that the waveforms $\psi_n(t)$, $n = 1, 2, 3$ are orthogonal we have to prove that

$$\int_{-\infty}^{\infty} \psi_m(t)\psi_n(t)dt = 0, \qquad m \neq n$$

Clearly,

$$\begin{aligned}
c_{12} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_2(t)dt = \int_0^4 \psi_1(t)\psi_2(t)dt \\
&= \int_0^2 \psi_1(t)\psi_2(t)dt + \int_2^4 \psi_1(t)\psi_2(t)dt \\
&= \frac{1}{4}\int_0^2 dt - \frac{1}{4}\int_2^4 dt = \frac{1}{4}\times 2 - \frac{1}{4}\times(4-2) \\
&= 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
c_{13} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_3(t)dt = \int_0^4 \psi_1(t)\psi_3(t)dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt - \frac{1}{4}\int_2^3 dt + \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
c_{23} &= \int_{-\infty}^{\infty} \psi_2(t)\psi_3(t)dt = \int_0^4 \psi_2(t)\psi_3(t)dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt + \frac{1}{4}\int_2^3 dt - \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

Thus, the signals $\psi_n(t)$ are orthogonal.

2) We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt, \qquad n = 1, 2, 3$$

$$\begin{aligned}
x_1 &= \int_0^4 x(t)\psi_1(t)dt = -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \\
x_2 &= \int_0^4 x(t)\psi_2(t)dt = \frac{1}{2}\int_0^4 x(t)dt = 0 \\
x_3 &= \int_0^4 x(t)\psi_3(t)dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0
\end{aligned}$$

As it is observed, $x(t)$ is orthogonal to the signal waveforms $\psi_n(t)$, $n = 1, 2, 3$ and thus it can not represented as a linear combination of these functions.

Problem 8.2

1) The expansion coefficients $\{c_n\}$, that minimize the mean square error, satisfy

$$c_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt = \int_0^4 \sin\frac{\pi t}{4}\psi_n(t)dt$$

Hence,

$$\begin{aligned} c_1 &= \int_0^4 \sin\frac{\pi t}{4}\psi_1(t)dt = \frac{1}{2}\int_0^2 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_2^4 \sin\frac{\pi t}{4}dt \\ &= -\frac{2}{\pi}\cos\frac{\pi t}{4}\bigg|_0^2 + \frac{2}{\pi}\cos\frac{\pi t}{4}\bigg|_2^4 \\ &= -\frac{2}{\pi}(0-1) + \frac{2}{\pi}(-1-0) = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} c_2 &= \int_0^4 \sin\frac{\pi t}{4}\psi_2(t)dt = \frac{1}{2}\int_0^4 \sin\frac{\pi t}{4}dt \\ &= -\frac{2}{\pi}\cos\frac{\pi t}{4}\bigg|_0^4 = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi} \end{aligned}$$

and

$$\begin{aligned} c_3 &= \int_0^4 \sin\frac{\pi t}{4}\psi_3(t)dt \\ &= \frac{1}{2}\int_0^1 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_1^2 \sin\frac{\pi t}{4}dt + \frac{1}{2}\int_2^3 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_3^4 \sin\frac{\pi t}{4}dt \\ &= 0 \end{aligned}$$

Note that c_1, c_2 can be found by inspection since $\sin\frac{\pi t}{4}$ is even with respect to the $x = 2$ axis and $\psi_1(t)$, $\psi_3(t)$ are odd with respect to the same axis.

2) The residual mean square error $E_{\min}$ can be found from

$$E_{\min} = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{3} |c_i|^2$$

Thus,

$$\begin{aligned} E_{\min} &= \int_0^4 \left(\sin\frac{\pi t}{4}\right)^2 dt - \left(\frac{4}{\pi}\right)^2 = \frac{1}{2}\int_0^4 \left(1 - \cos\frac{\pi t}{2}\right) dt - \frac{16}{\pi^2} \\ &= 2 - \frac{1}{\pi}\sin\frac{\pi t}{2}\bigg|_0^4 - \frac{16}{\pi^2} = 2 - \frac{16}{\pi^2} \end{aligned}$$

Problem 8.3

1) As an orthonormal set of basis functions we consider the set

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$\psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad \psi_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

2) The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

3) The distance between the first and the second vector is

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

Similarly we find that

$$\begin{aligned} d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}\right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}\right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix}\right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix}\right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix}\right|^2} = \sqrt{19} \end{aligned}$$

forms is 4

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 8.4

As a set of orthonormal functions we consider the waveforms

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \quad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases} \quad \psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

Problem 8.5

1) The impulse response of the filter matched to $s(t)$ is

$$h(t) = s(T - t) = s(3 - t) = s(t)$$

where we have used the fact that $s(t)$ is even with respect to the $t = \frac{T}{2} = \frac{3}{2}$ axis.

2) The output of the matched filter is

$$\begin{aligned} y(t) &= s(t) \star s(t) = \int_0^t s(\tau)s(t-\tau)d\tau \\ &= \begin{cases} 0 & t < 0 \\ A^2 t & 0 \le t < 1 \\ A^2(2-t) & 1 \le t < 2 \\ 2A^2(t-2) & 2 \le t < 3 \\ 2A^2(4-t) & 3 \le t < 4 \\ A^2(t-4) & 4 \le t < 5 \\ A^2(6-t) & 5 \le t < 6 \\ 0 & 6 \le t \end{cases} \end{aligned}$$

A sketch of $y(t)$ is depicted in the next figure

For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \Longrightarrow A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.

Problem 8.17

The energy of the two signals $s_1(t)$ and $s_2(t)$ is

$$\mathcal{E}_b = A^2T$$

The dimensionality of the signal space is one, and by choosing the basis function as

$$\psi(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \leq t < \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} \leq t \leq T \end{cases}$$

we find the vector representation of the signals as

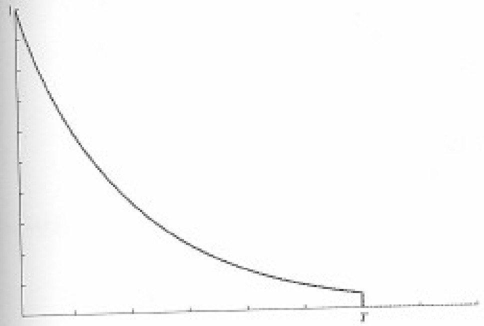
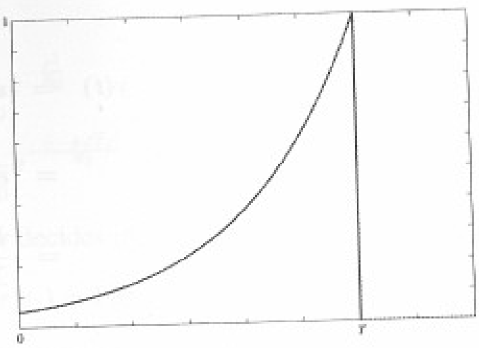
$$s_{1,2} = \pm A\sqrt{T} + n$$

with n a zero-mean Gaussian random variable of variance $\frac{N_0}{2}$. The probability of error for antipodal signals is given by, where $\mathcal{E}_b = A^2T$. Hence,

$$P(e) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left[\sqrt{\frac{2A^2T}{N_0}}\right]$$

Problem 8.18

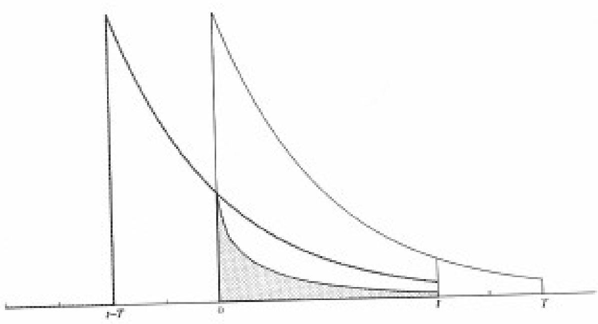
Plots of $s(t)$ and $h(t)$ are shown on left and right, respectively.

The output of the matched filter is

$$y(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)\,d\tau$$

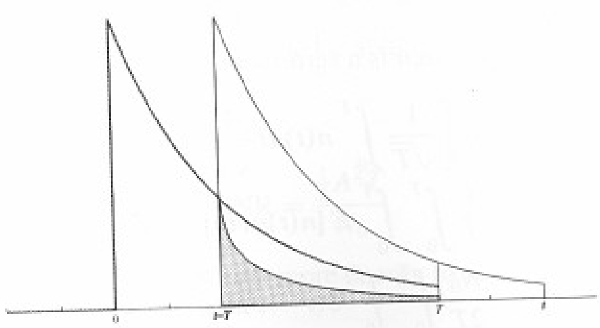
For $t < 0$, there is no overlap and the integral is zero. For $0 < t \le T$ we have the following figure, where the the product of the two signals in the overlapping region is $s(\tau)h(t-\tau) = e^{-\tau} \times e^{-(T-t+\tau)} = e^{t-T-2\tau}$ and the integral is the area of the shaded region.



For this case we have

$$\begin{aligned} y(t) &= e^{t-T}\int_0^t e^{-2\tau}\,d\tau \\ &= e^{t-T}\left[-\frac{1}{2}e^{-2\tau}\right]_0^t \\ &= \frac{1}{2}e^{-T}\left(e^{t}-e^{-t}\right) \end{aligned}$$

For $T < t \le 2T$ we have the following figure



and

$$\begin{aligned} y(t) &= e^{t-T}\int_{t-T}^{T} e^{-2\tau}\,d\tau \\ &= e^{t-T}\left[-\frac{1}{2}e^{-2\tau}\right]_{t-T}^{T} \\ &= \frac{1}{2}e^{-t+T}-\frac{1}{2}e^{t-3T} \end{aligned}$$

Therefore

$$y(t)=\begin{cases} \frac{1}{2}e^{-T}\left(e^{t}-e^{-t}\right) & 0<t\leq T \\ \frac{1}{2}e^{-t+T}-\frac{1}{2}e^{t-3T} & T<t\leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Problem 8.19

We have $P_{av}=R\mathcal{E}_b=2\times 10^{6}\mathcal{E}_b$, hence

$$P_b=Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)=Q\left(\sqrt{\frac{2P_{av}}{RN_0}}\right)=10^{-6}$$

Using the Q-function table (page 220) we have $Q(4.77)\approx 10^{-6}$, therefore

$$\sqrt{\frac{2P_{av}}{RN_0}}=\sqrt{\frac{2P_{av}}{2\times 10^{6}N_0}}=Q(4.77)\Rightarrow\frac{P_{av}}{10^{6}N_0}=4.77^{2}$$

From this we have $\frac{P_{av}}{N_0}=22.753\times 10^{6}$.

Problem 8.20

a) The received signal may be expressed as

$$r(t)=\begin{cases} n(t) & \text{if } s_0(t) \text{ was transmitted} \\ A+n(t) & \text{if } s_1(t) \text{ was transmitted} \end{cases}$$

Assuming that $s(t)$ has unit energy, then the sampled outputs of the crosscorrelators are

$$r=s_m+n,\qquad m=0,1$$

where $s_0=0$, $s_1=A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance

$$\begin{aligned} \sigma_n^2 &= E\left[\frac{1}{\sqrt{T}}\int_0^T n(t)dt\frac{1}{\sqrt{T}}\int_0^T n(\tau)d\tau\right] \\ &= \frac{1}{T}\int_0^T\int_0^T E\left[n(t)n(\tau)\right]dtd\tau \\ &= \frac{N_0}{2T}\int_0^T\int_0^T \delta(t-\tau)dtd\tau=\frac{N_0}{2} \end{aligned}$$

Problem 8.21

1) Since $m_2(t) = -m_3(t)$ the dimensionality of the signal space is two.

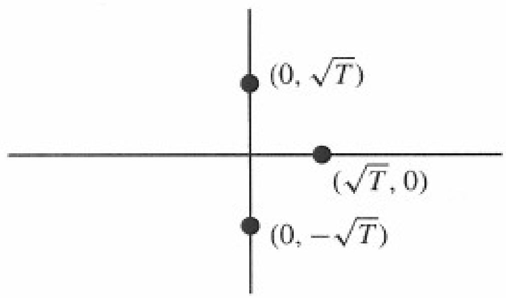
2) As a basis of the signal space we consider the functions

$$\psi_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \le T \\ 0 & \text{otherwise} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned} \mathbf{m}_1 &= [\sqrt{T},\ 0] \\ \mathbf{m}_2 &= [0,\ \sqrt{T}] \\ \mathbf{m}_3 &= [0,\ -\sqrt{T}] \end{aligned}$$

3) The signal constellation is depicted in the next figure



4) The three possible outputs of the matched filters, corresponding to the three possible transmitted signals are $(r_1, r_2) = (\sqrt{T} + n_1, n_2)$, $(n_1, \sqrt{T} + n_2)$ and $(n_1, -\sqrt{T} + n_2)$, where n_1, n_2 are zero-mean Gaussian random variables with variance $\frac{N_0}{2}$. If all the signals are equiprobable the optimum decision rule selects the signal that maximizes the metric

$$C(\mathbf{r} \cdot \mathbf{m}_i) = 2\mathbf{r} \cdot \mathbf{m}_i - |\mathbf{m}_i|^2$$

or since $|\mathbf{m}_i|^2$ is the same for all i,

$$C'(\mathbf{r} \cdot \mathbf{m}_i) = \mathbf{r} \cdot \mathbf{m}_i$$

Thus the optimal decision region R_1 for $\mathbf{m}_1$ is the set of points (r_1, r_2), such that $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_2$ and $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_3$. Since $(r_1, r_2) \cdot \mathbf{m}_1 = \sqrt{T} r_1$, $(r_1, r_2) \cdot \mathbf{m}_2 = \sqrt{T} r_2$ and $(r_1, r_2) \cdot \mathbf{m}_3 = -\sqrt{T} r_2$, the previous conditions are written as

$$r_1 > r_2 \quad \text{and} \quad r_1 > -r_2$$

Similarly we find that R_2 is the set of points (r_1, r_2) that satisfy $r_2 > 0$, $r_2 > r_1$ and R_3 is the region such that $r_2 < 0$ and $r_2 < -r_1$. The regions R_1, R_2 and R_3 are shown in the next figure.

and therefore, the Fourier transform of the signal matched to $s(t)$ is

$$
\begin{aligned}
H(f) &= S^*(f)e^{-j2\pi fT} = S^*(f)e^{-j2\pi fnT_c} \\
&= P^*(f)\sum_{k=1}^{n} c_k e^{j2\pi fkT_c} e^{-j2\pi fnT_c} \\
&= P^*(f)\sum_{i=1}^{n} c_{n-i+1} e^{-j2\pi f(i-1)T_c} \\
&= P^*(f)\mathcal{F}[g(t)]
\end{aligned}
$$

Thus, the matched filter $H(f)$ can be considered as the cascade of a filter,with impulse response $p(-t)$, matched to the pulse $p(t)$ and a filter, with impulse response $g(t)$, matched to the signal $y(t) = \sum_{k=1}^{n} c_k\delta(t-kT_c)$. The output of the matched filter at $t = nT_c$ is

$$
\begin{aligned}
\int_{-\infty}^{\infty} |s(t)|^2 &= \sum_{k=1}^{n} c_k^2 \int_{-\infty}^{\infty} p^2(t-kT_c)dt \\
&= T_c \sum_{k=1}^{n} c_k^2
\end{aligned}
$$

where we have used the fact that $p(t)$ is a rectangular pulse of unit amplitude and duration T_c.

Problem 8.31

The bandwidth required for transmission of an M-ary PAM signal is

$$W = \frac{R_b}{2\log_2 M} \text{ Hz}$$

Since,

$$R_b = 8 \times 10^3 \frac{\text{samples}}{\text{sec}} \times 8 \frac{\text{bits}}{\text{sample}} = 64 \times 10^3 \frac{\text{bits}}{\text{sec}}$$

we obtain

$$W = \begin{cases} 16 \text{ KHz} & M = 4 \\ 10.667 \text{ KHz} & M = 8 \\ 8 \text{ KHz} & M = 16 \end{cases}$$

Problem 8.32

The vector $\mathbf{r} = [r_1, r_2]$ at the output of the integrators is

$$\mathbf{r} = [r_1, r_2] = [\int_0^{1.5} r(t)dt, \ \int_1^2 r(t)dt]$$