The modulating signal is $m(t) = \cos(2\pi 100t)$ whereas the carrier signal is $c(t) = 20\cos(2\pi 1000t)$.

2) Since $-1 \leq \cos(2\pi 100t) \leq 1$, we immediately have that the modulation index is $\alpha = \frac{1}{2}$.

3) The power of the carrier component is $P_{\text{carrier}} = \frac{400}{2} = 200$, whereas the power in the sidebands is $P_{\text{sidebands}} = \frac{400\alpha^2}{2} = 50$. Hence,

$$\frac{P_{\text{sidebands}}}{P_{\text{carrier}}} = \frac{50}{200} = \frac{1}{4}$$

Problem 3.15

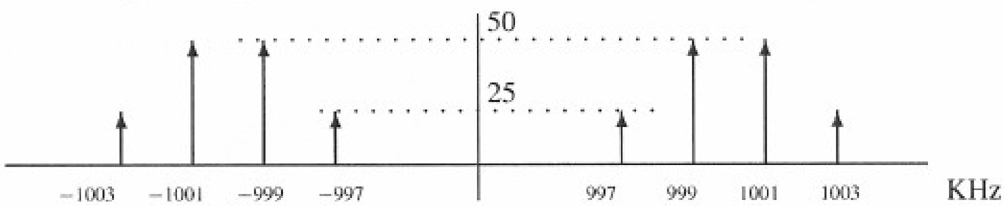
1) The modulated signal is written as

$$\begin{aligned} u(t) &= 100(2\cos(2\pi 10^3 t) + \cos(2\pi 3 \times 10^3 t))\cos(2\pi f_c t) \\ &= 200\cos(2\pi 10^3 t)\cos(2\pi f_c t) + 100\cos(2\pi 3 \times 10^3 t)\cos(2\pi f_c t) \\ &= 100\left[\cos(2\pi(f_c + 10^3)t) + \cos(2\pi(f_c - 10^3)t)\right] \\ &\quad +50\left[\cos(2\pi(f_c + 3 \times 10^3)t) + \cos(2\pi(f_c - 3 \times 10^3)t)\right] \end{aligned}$$

Taking the Fourier transform of the previous expression, we obtain

$$\begin{aligned} U(f) &= 50\left[\delta(f - (f_c + 10^3)) + \delta(f + f_c + 10^3)\right. \\ &+ \left.\delta(f - (f_c - 10^3)) + \delta(f + f_c - 10^3)\right] \\ &+ 25\left[\delta(f - (f_c + 3 \times 10^3)) + \delta(f + f_c + 3 \times 10^3)\right. \\ &+ \left.\delta(f - (f_c - 3 \times 10^3)) + \delta(f + f_c - 3 \times 10^3)\right] \end{aligned}$$

The spectrum of the signal is depicted in the next figure



2) The average power in the frequencies $f_c + 1000$ and $f_c - 1000$ is

$$P_{f_c+1000} = P_{f_c-1000} = \frac{100^2}{2} = 5000$$

The average power in the frequencies $f_c + 3000$ and $f_c - 3000$ is

$$P_{f_c+3000} = P_{f_c-3000} = \frac{50^2}{2} = 1250$$

Problem 3.16

1) The Hilbert transform of $\cos(2\pi 1000t)$ is $\sin(2\pi 1000t)$, whereas the Hilbert transform of$\sin(2\pi 1000t)$ is $-\cos(2\pi 1000t)$. Thus

$$\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$$

2) The expression for the LSSB AM signal is

$$u_l(t) = A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$$

Substituting $A_c = 100$, $m(t) = \cos(2\pi 1000t) + 2\sin(2\pi 1000t)$ and $\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$ in the previous, we obtain

$$\begin{aligned}
u_l(t) &= 100\left[\cos(2\pi 1000t) + 2\sin(2\pi 1000t)\right]\cos(2\pi f_c t) \\
&+ 100\left[\sin(2\pi 1000t) - 2\cos(2\pi 1000t)\right]\sin(2\pi f_c t) \\
&= 100\left[\cos(2\pi 1000t)\cos(2\pi f_c t) + \sin(2\pi 1000t)\sin(2\pi f_c t)\right] \\
&+ 200\left[\cos(2\pi f_c t)\sin(2\pi 1000t) - \sin(2\pi f_c t)\cos(2\pi 1000t)\right] \\
&= 100\cos(2\pi(f_c - 1000)t) - 200\sin(2\pi(f_c - 1000)t)
\end{aligned}$$

3) Taking the Fourier transform of the previous expression we obtain

$$\begin{aligned}
U_l(f) &= 50\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\
&+ 100j\left(\delta(f - f_c + 1000) - \delta(f + f_c - 1000)\right) \\
&= (50 + 100j)\delta(f - f_c + 1000) + (50 - 100j)\delta(f + f_c - 1000)
\end{aligned}$$

Hence, the magnitude spectrum is given by

$$\begin{aligned}
|U_l(f)| &= \sqrt{50^2 + 100^2}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\
&= 10\sqrt{125}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right)
\end{aligned}$$

Problem 3.17

The input to the upper LPF is

$$\begin{aligned}
u_u(t) &= \cos(2\pi f_m t)\cos(2\pi f_1 t) \\
&= \frac{1}{2}\left[\cos(2\pi(f_1 - f_m)t) + \cos(2\pi(f_1 + f_m)t)\right]
\end{aligned}$$

whereas the input to the lower LPF is

$$\begin{aligned}
u_l(t) &= \cos(2\pi f_m t)\sin(2\pi f_1 t) \\
&= \frac{1}{2}\left[\sin(2\pi(f_1 - f_m)t) + \sin(2\pi(f_1 + f_m)t)\right]
\end{aligned}$$

If we select f_1 such that $|f_1 - f_m| < W$ and $f_1 + f_m > W$, then the two lowpass filters will cut-off the frequency components outside the interval $[-W, W]$, so that the output of the upper and lower LPF is

$$\begin{aligned}
y_u(t) &= \cos(2\pi(f_1 - f_m)t) \\
y_l(t) &= \sin(2\pi(f_1 - f_m)t)
\end{aligned}$$

Since $A_m = \max[|m(t)|]$ we conclude that the modulation index is

$$\alpha = \frac{2bA_m}{a}$$

Problem 3.20

1) When USSB is employed the bandwidth of the modulated signal is the same with the bandwidth of the message signal. Hence,

$$W_{\mathrm{USSB}} = W = 10^4 \text{ Hz}$$

2) When DSB is used, then the bandwidth of the transmitted signal is twice the bandwidth of the message signal. Thus,

$$W_{\mathrm{DSB}} = 2W = 2 \times 10^4 \text{ Hz}$$

3) If conventional AM is employed, then

$$W_{\mathrm{AM}} = 2W = 2 \times 10^4 \text{ Hz}$$

4) Using Carson's rule, the effective bandwidth of the FM modulated signal is

$$B_c = (2\beta + 1)W = 2\left(\frac{k_f \max[|m(t)|]}{W} + 1\right) W = 2(k_f + W) = 140000 \text{ Hz}$$

Problem 3.21

1) The lowpass equivalent transfer function of the system is

$$H_l(f) = 2u_{-1}(f + f_c)H(f + f_c) = 2\begin{cases} \frac{1}{W}f + \frac{1}{2} & |f| \leq \frac{W}{2} \\ 1 & \frac{W}{2} < f \leq W \end{cases}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned}
h_l(t) &= \mathcal{F}^{-1}[H_l(f)] = \int_{-\frac{W}{2}}^{W} H_l(f)e^{j2\pi f t}df \\
&= 2\int_{-\frac{W}{2}}^{\frac{W}{2}} (\frac{1}{W}f + \frac{1}{2})e^{j2\pi f t}df + 2\int_{\frac{W}{2}}^{W} e^{j2\pi f t}df \\
&= \frac{2}{W}\left(\frac{1}{j2\pi t}fe^{j2\pi f t} + \frac{1}{4\pi^2 t^2}e^{j2\pi f t}\right)\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{1}{j2\pi t}e^{j2\pi f t}\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{2}{j2\pi t}e^{j2\pi f t}\Bigg|_{\frac{W}{2}}^{W} \\
&= \frac{1}{j\pi t}e^{j2\pi W t} + \frac{j}{\pi^2 t^2 W}\sin(\pi W t) \\
&= \frac{j}{\pi t}\left[\mathrm{sinc}(Wt) - e^{j2\pi W t}\right]
\end{aligned}$$

Solution to Chapter 4 Problems

Problem 4.1

1) Since $\mathcal{F}[\text{sinc}(400t)] = \frac{1}{400}\Pi(\frac{f}{400})$, the bandwidth of the message signal is $W = 200$ and the resulting modulation index

$$\beta_f = \frac{k_f \max[|m(t)|]}{W} = \frac{k_f 10}{W} = 6 \Longrightarrow k_f = 120$$

Hence, the modulated signal is

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) \\ &= 100\cos(2\pi f_c t + +2\pi 1200 \int_{-\infty}^{t} \text{sinc}(400\tau)d\tau) \end{aligned}$$

2) The maximum frequency deviation of the modulated signal is

$$\Delta f_{\max} = \beta_f W = 6 \times 200 = 1200$$

3) Since the modulated signal is essentially a sinusoidal signal with amplitude $A = 100$, we have

$$P = \frac{A^2}{2} = 5000$$

4) Using Carson's rule, the effective bandwidth of the modulated signal can be approximated by

$$B_c = 2(\beta_f + 1)W = 2(6+1)200 = 2800 \text{ Hz}$$

Problem 4.2

1) The maximum phase deviation of the PM signal is

$$\Delta\phi_{\max} = k_p \max[|m(t)|] = k_p$$

The phase of the FM modulated signal is

$$\begin{aligned} \phi(t) &= 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau = 2\pi k_f \int_{0}^{t} m(\tau)d\tau \\ &= \begin{cases} 2\pi k_f \int_0^t \tau d\tau = \pi k_f t^2 & 0 \le t < 1 \\ \pi k_f + 2\pi k_f \int_1^t d\tau = \pi k_f + 2\pi k_f (t-1) & 1 \le t < 2 \\ \pi k_f + 2\pi k_f - 2\pi k_f \int_2^t d\tau = 3\pi k_f - 2\pi k_f (t-2) & 2 \le t < 3 \\ \pi k_f & 3 \le t \end{cases} \end{aligned}$$

The maximum value of $\phi(t)$ is achieved for $t = 2$ and is equal to $3\pi k_f$. Thus, the desired relation between k_p and k_f is

$$k_p = 3\pi k_f$$

2) The instantaneous frequency for the PM modulated signal is

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi}k_p\frac{d}{dt}m(t)$$

For the $m(t)$ given in Fig. P-4.2, the maximum value of $\frac{d}{dt}m(t)$ is achieved for t in [0, 1] and it is equal to one. Hence,

$$\max(f_i(t)) = f_c + \frac{1}{2\pi}$$

For the FM signal $f_i(t) = f_c + k_f m(t)$. Thus, the maximum instantaneous frequency is

$$\max(f_i(t)) = f_c + k_f = f_c + 1$$

Problem 4.3

For an angle modulated signal we have $x(t) = A_c\cos(2\pi f_c t + \phi(t))$, therefore The lowpass equivalent of the signal is $x_l(t) = A_c e^{j\phi(t)}$ with Envelope A_c and phase $\pi(t)$ and in phase an quadrature components $A_c\cos(\phi(t))$ and $A_c\sin(\phi(t))$, respectively. Hence we have the following

$$\text{PM}\begin{cases} A_c & \text{envelope} \\ k_p m(t) & \text{phase} \\ A_c\cos\left(k_p m(t)\right) & \text{in-phase comp.} \\ A_c\sin\left(k_p m(t)\right) & \text{quadrature comp.} \end{cases} \qquad \text{FM}\begin{cases} A_c & \text{envelope} \\ 2\pi k_f\int_{-\infty}^{t} m(\tau)\,d\tau & \text{phase} \\ A_c\cos\left(2\pi k_f\int_{-\infty}^{t} m(\tau)\,d\tau\right) & \text{in-phase comp.} \\ A_c\sin\left(2\pi k_f\int_{-\infty}^{t} m(\tau)\,d\tau\right) & \text{quadrature comp.} \end{cases}$$

Problem 4.4

1) Since an angle modulated signal is essentially a sinusoidal signal with constant amplitude, we have

$$P = \frac{A_c^2}{2} \Longrightarrow P = \frac{100^2}{2} = 5000$$

The same result is obtained if we use the expansion

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

along with the identity

$$J_0^2(\beta) + 2\sum_{n=1}^{\infty} J_n^2(\beta) = 1$$

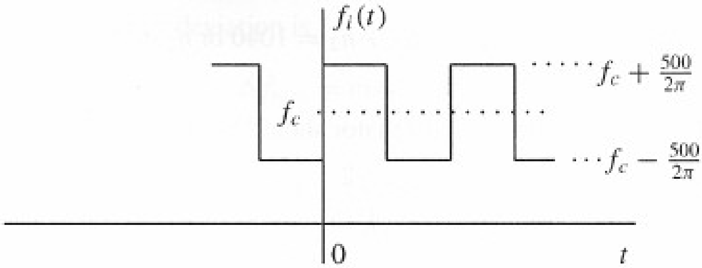
where $\phi_l = 0$ for $l \geq 0$ and $\phi_l = \pi$ for negative values of l.

Problem 4.8

1) The instantaneous frequency is given by

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi}100m(t)$$

A plot of $f_i(t)$ is given in the next figure



2) The peak frequency deviation is given by

$$\Delta f_{\max} = k_f \max[|m(t)|] = \frac{100}{2\pi}5 = \frac{250}{\pi}$$

Problem 4.9

1) The modulation index is

$$\beta = \frac{k_f \max[|m(t)|]}{f_m} = \frac{\Delta f_{\max}}{f_m} = \frac{20 \times 10^3}{10^4} = 2$$

The modulated signal $u(t)$ has the form

$$\begin{aligned} u(t) &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t + \phi_n) \\ &= \sum_{n=-\infty}^{\infty} 100 J_n(2)\cos(2\pi(10^8 + n10^4)t + \phi_n) \end{aligned}$$

The power of the unmodulated carrier signal is $P = \frac{100^2}{2} = 5000$. The power in the frequency component $f = f_c + k10^4$ is

$$P_{f_c + kf_m} = \frac{100^2 J_k^2(2)}{2}$$

The next table shows the values of $J_k(2)$, the frequency $f_c + kf_m$, the amplitude $100J_k(2)$ and the power $P_{f_c+kf_m}$ for various values of k.

Index k	$J_k(2)$	Frequency Hz	Amplitude $100J_k(2)$	Power $P_{f_c+kf_m}$
0	.2239	10^8	22.39	250.63
1	.5767	$10^8 + 10^4$	57.67	1663.1
2	.3528	$10^8 + 2 \times 10^4$	35.28	622.46
3	.1289	$10^8 + 3 \times 10^4$	12.89	83.13
4	.0340	$10^8 + 4 \times 10^4$	3.40	5.7785

As it is observed from the table the signal components that have a power level greater than 500 (= 10% of the power of the unmodulated signal) are those with frequencies $10^8 + 10^4$ and $10^8 + 2 \times 10^4$. Since $J_n^2(\beta) = J_{-n}^2(\beta)$ it is conceivable that the signal components with frequency $10^8 - 10^4$ and $10^8 - 2 \times 10^4$ will satisfy the condition of minimum power level. Hence, there are four signal components that have a power of at least 10% of the power of the unmodulated signal. The components with frequencies $10^8 + 10^4$, $10^8 - 10^4$ have an amplitude equal to 57.67, whereas the signal components with frequencies $10^8 + 2 \times 10^4$, $10^8 - 2 \times 10^4$ have an amplitude equal to 35.28.

2) Using Carson's rule, the approximate bandwidth of the FM signal is

$$B_c = 2(\beta + 1) f_m = 2(2 + 1)10^4 = 6 \times 10^4 \text{ Hz}$$

Problem 4.10

1)

$$\begin{aligned} \beta_p &= k_p \max[|m(t)|] = 1.5 \times 2 = 3 \\ \beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000 \times 2}{1000} = 6 \end{aligned}$$

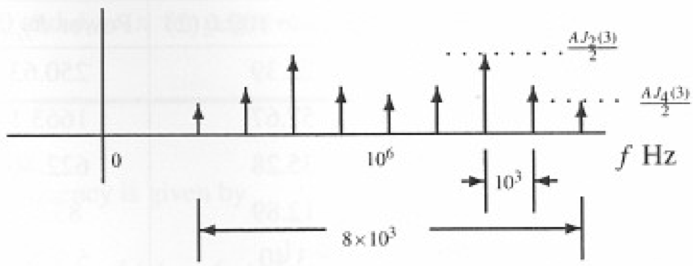
2) Using Carson's rule we obtain

$$\begin{aligned} B_{\text{PM}} &= 2(\beta_p + 1) f_m = 8 \times 1000 = 8000 \\ B_{\text{FM}} &= 2(\beta_f + 1) f_m = 14 \times 1000 = 14000 \end{aligned}$$

3) The PM modulated signal can be written as

$$u(t) = \sum_{n=-\infty}^{\infty} A J_n(\beta_p) \cos(2\pi(10^6 + n10^3)t)$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 4 \times 10^3, 10^6 + 4 \times 10^3]$. Note that $J_0(3) = -.2601$, $J_1(3) = 0.3391$, $J_2(3) = 0.4861$, $J_3(3) = 0.3091$ and $J_4(3) = 0.1320$.

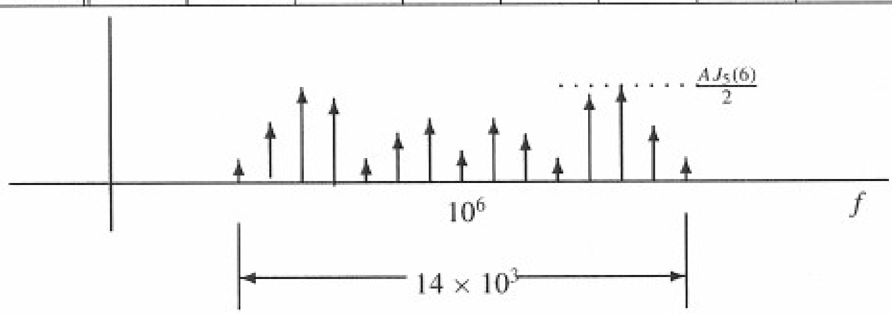


In the case of the FM modulated signal

$$
\begin{aligned}
u(t) &= A\cos(2\pi f_c t + \beta_f \sin(2000\pi t)) \\
&= \sum_{n=-\infty}^{\infty} A J_n(6)\cos(2\pi(10^6 + n10^3)t + \phi_n)
\end{aligned}
$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 7\times 10^3, 10^6 - 7\times 10^3]$. The values of $J_n(6)$ for $n = 0, \ldots, 7$ are given in the following table.

n	0	1	2	3	4	5	6	7
$J_n(6)$	.1506	-.2767	-.2429	.1148	.3578	.3621	.2458	.1296



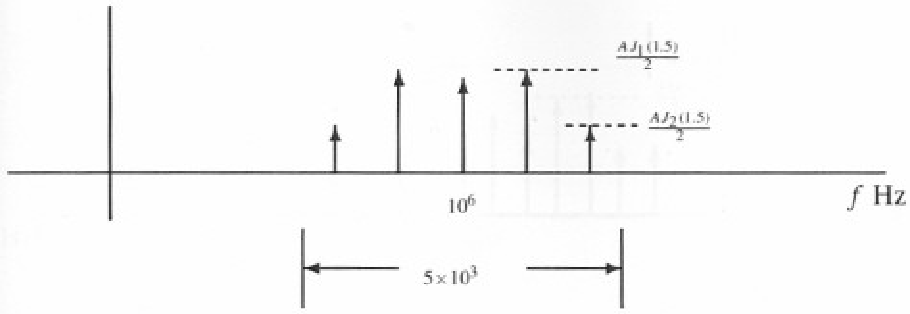
4) If the amplitude of $m(t)$ is decreased by a factor of two, then $m(t) = \cos(2\pi 10^3 t)$ and

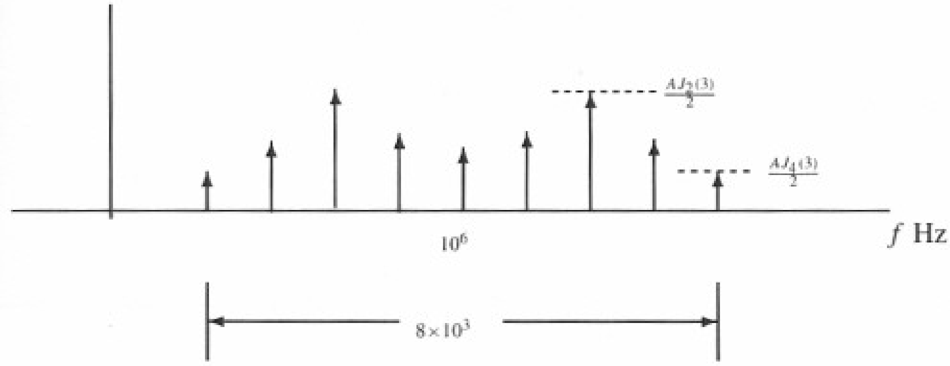
$$
\begin{aligned}
\beta_p &= k_p \max[|m(t)|] = 1.5 \\
\beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000}{1000} = 3
\end{aligned}
$$

The bandwidth is determined using Carson's rule as

$$
\begin{aligned}
B_{\text{PM}} &= 2(\beta_p + 1) f_m = 5 \times 1000 = 5000 \\
B_{\text{FM}} &= 2(\beta_f + 1) f_m = 8 \times 1000 = 8000
\end{aligned}
$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that $J_0(1.5) = .5118$, $J_1(1.5) = .5579$ and $J_2(1.5) = .2321$.





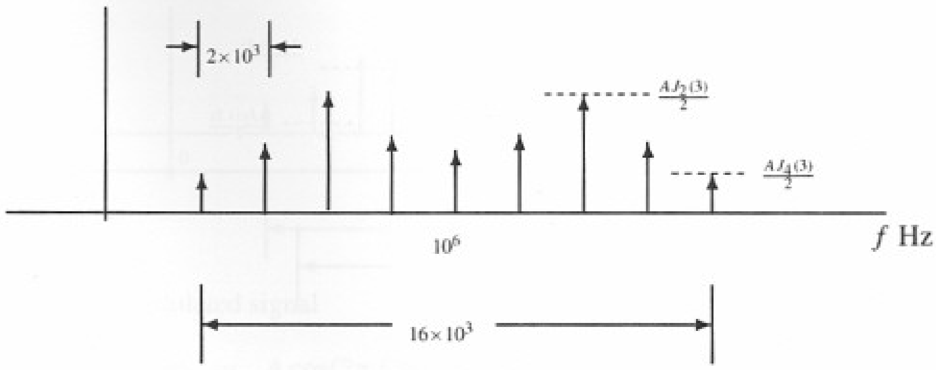
5) If the frequency of $m(t)$ is increased by a factor of two, then $m(t) = 2\cos(2\pi 2 \times 10^3 t)$ and

$$\begin{aligned}
\beta_p &= k_p \max[|m(t)|] = 1.5 \times 2 = 3 \\
\beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000 \times 2}{2000} = 3
\end{aligned}$$

The bandwidth is determined using Carson's rule as

$$\begin{aligned}
B_{\text{PM}} &= 2(\beta_p + 1) f_m = 8 \times 2000 = 16000 \\
B_{\text{FM}} &= 2(\beta_f + 1) f_m = 8 \times 2000 = 16000
\end{aligned}$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that doubling the frequency has no effect on the number of harmonics in the bandwidth of the PM signal, whereas it decreases the number of harmonics in the bandwidth of the FM signal from 14 to 8.



Problem 4.11

1) The PM modulated signal is

$$
\begin{aligned}
u(t) &= 100\cos(2\pi f_c t + \frac{\pi}{2}\cos(2\pi 1000t)) \\
&= \sum_{n=-\infty}^{\infty} 100 J_n\left(\frac{\pi}{2}\right)\cos(2\pi(10^8 + n10^3)t)
\end{aligned}
$$

The next table tabulates $J_n(\beta)$ for $\beta = \frac{\pi}{2}$ and $n = 0, \ldots, 4$.

n	0	1	2	3	4
$J_n(\beta)$	.4720	.5668	.2497	.0690	.0140

The total power of the modulated signal is $P_{\text{tot}} = \frac{100^2}{2} = 5000$. To find the effective bandwidth of the signal we calculate the index k such that

$$
\sum_{n=-k}^{k} \frac{100^2}{2} J_n^2\left(\frac{\pi}{2}\right) \geq 0.99 \times 5000 \Longrightarrow \sum_{n=-k}^{k} J_n^2\left(\frac{\pi}{2}\right) \geq 0.99
$$

By trial end error we find that the smallest index k is 2. Hence the effective bandwidth is

$$
B_{\text{eff}} = 4 \times 10^3 = 4000
$$

In the the next figure we sketch the magnitude spectrum for the positive frequencies.

Solution to Chapter 7 Problems

Problem 7.1

1. Since the maximum frequency in $X(f)$ is is 40 KHz, the minimum sampling rate is $f_s = 2W = 80$ KHz.

2. Here $f_s = 2W + W_G = 2 \times 40 + 10 = 90$ KHz.

3. $X_1(f) = \frac{1}{2}X(f-40000)+\frac{1}{2}X(f+40000)$, the maximum frequency in $X_1(f)$ is 40000+40000=80000 Hz, and the minimum sampling rate is $f_s = 2 \times 80000 = 160000$. From this the maximum sampling interval is $T_s = 1/f_s = 1/160000 = 6.25\mu$ sec.

Problem 7.2

For no aliasing to occur we must sample at the Nyquist rate

$$f_s = 2 \cdot 6000 \text{ samples/sec } = 12000 \text{ samples/sec}$$

With a guard band of 2000

$$f_s - 2W = 2000 \Longrightarrow f_s = 14000$$

The reconstruction filter should not pick-up frequencies of the images of the spectrum $X(f)$. The nearest image spectrum is centered at f_s and occupies the frequency band $[f_s - W, f_s + W]$. Thus the highest frequency of the reconstruction filter (= 10000) should satisfy

$$10000 \leq f_s - W \Longrightarrow f_s \geq 16000$$

For the value $f_s = 16000$, K should be such that

$$K \cdot f_s = 1 \Longrightarrow K = (16000)^{-1}$$

Problem 7.3

$$x(t) = A\text{sinc}(1000\pi t) \Longrightarrow X(f) = \frac{A}{1000}\Pi(\frac{f}{1000})$$

Thus the bandwidth W of $x(t)$ is $1000/2 = 500$. Since we sample at $f_s = 2000$ there is a gap between the image spectra equal to

$$2000 - 500 - W = 1000$$

Problem 7.12

1) The area between the two squares is $4\times 4 - 2\times 2 = 12$. Hence, $f_{X,Y}(x,y) = \frac{1}{12}$. The marginal probability $f_X(x)$ is given by $f_X(x) = \int_{-2}^{2} f_{X,Y}(x,y)dy$. If $-2 \leq X < -1$, then

$$f_X(x) = \int_{-2}^{2} f_{X,Y}(x,y)dy = \frac{1}{12}y\Big|_{-2}^{2} = \frac{1}{3}$$

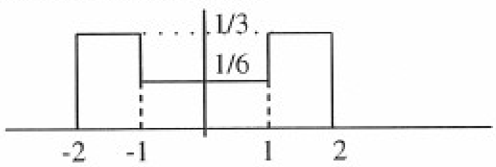
If $-1 \leq X < 1$, then

$$f_X(x) = \int_{-2}^{-1} \frac{1}{12}dy + \int_{1}^{2} \frac{1}{12}dy = \frac{1}{6}$$

Finally, if $1 \leq X \leq 2$, then

$$f_X(x) = \int_{-2}^{2} f_{X,Y}(x,y)dy = \frac{1}{12}y\Big|_{-2}^{2} = \frac{1}{3}$$

The next figure depicts the marginal distribution $f_X(x)$.



Similarly we find that

$$f_Y(y) = \begin{cases} \frac{1}{3} & -2 \leq y < -1 \\ \frac{1}{6} & -1 \leq y < -1 \\ \frac{1}{3} & 1 \leq y \leq 2 \end{cases}$$

2) The quantization levels $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ and $\hat{x}_4$ are set to $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$ respectively. The resulting distortion is

$$\begin{aligned} D_X &= 2\int_{-2}^{-1}(x+\frac{3}{2})^2 f_X(x)dx + 2\int_{-1}^{0}(x+\frac{1}{2})^2 f_X(x)dx \\ &= \frac{2}{3}\int_{-2}^{-1}(x^2+3x+\frac{9}{4})dx + \frac{2}{6}\int_{-1}^{0}(x^2+x+\frac{1}{4})dx \\ &= \frac{2}{3}\left(\frac{1}{3}x^3+\frac{3}{2}x^2+\frac{9}{4}x\right)\Big|_{-2}^{-1} + \frac{2}{6}\left(\frac{1}{3}x^3+\frac{1}{2}x^2+\frac{1}{4}x\right)\Big|_{-1}^{0} \\ &= \frac{1}{12} \end{aligned}$$

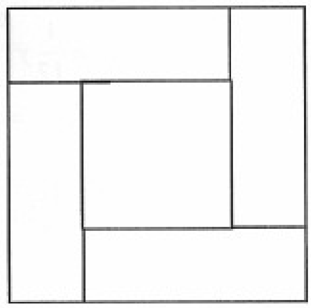
The total distortion is

$$D_{\text{total}} = D_X + D_Y = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

whereas the resulting number of bits per (X, Y) pair

$$R = R_X + R_Y = \log_2 4 + \log_2 4 = 4$$

3) Suppose that we divide the region over which $p(x, y) \neq 0$ into L equal subregions. The case of $L = 4$ is depicted in the next figure.



For each subregion the quantization output vector $(\hat{x}, \hat{y})$ is the centroid of the corresponding rectangle. Since, each subregion has the same shape (uniform quantization), a rectangle with width equal to one and length $12/L$, the distortion of the vector quantizer is

$$\begin{aligned} D &= \int_0^1 \int_0^{\frac{12}{L}} [(x, y) - (\frac{1}{2}, \frac{12}{2L})]^2 \frac{L}{12} dxdy \\ &= \frac{L}{12} \int_0^1 \int_0^{\frac{12}{L}} \left[(x - \frac{1}{2})^2 + (y - \frac{12}{2L})^2 \right] dxdy \\ &= \frac{L}{12} \left[\frac{12}{L} \frac{1}{12} + \frac{12^3}{L^3} \frac{1}{12} \right] = \frac{1}{12} + \frac{12}{L^2} \end{aligned}$$

If we set $D = \frac{1}{6}$, we obtain

$$\frac{12}{L^2} = \frac{1}{12} \Longrightarrow L = \sqrt{144} = 12$$

Thus, we have to divide the area over which $p(x, y) \neq 0$, into 12 equal subregions in order to achieve the same distortion. In this case the resulting number of bits per source output pair (X, Y) is $R = \log_2 12 = 3.585$.

Problem 7.13

1) The joint probability density function is $f_{XY}(x, y) = \frac{1}{(2\sqrt{2})^2} = \frac{1}{8}$. The marginal distribution $f_X(x)$ is $f_X(x) = \int_y f_{XY}(x, y) dy$. If $-2 \leq x \leq 0$,then

$$f_X(x) = \int_{-x-2}^{x+2} f_{X,Y}(x, y) dy = \frac{1}{8} y \Big|_{-x-2}^{x+2} = \frac{x+2}{4}$$

If $0 \leq x \leq 2$,then

$$f_X(x) = \int_{x-2}^{-x+2} f_{X,Y}(x, y) dy = \frac{1}{8} y \Big|_{x-2}^{-x+2} = \frac{-x+2}{4}$$

The next figure depicts $f_X(x)$.

3. The minimum transmission bandwidth is obtained from $\text{BW} = \nu W = 12 \times 200 = 2400$ Hz.

Problem 7.20

1. Any probability density function satisfies $\int_{-\infty}^{\infty} f(x)\,dx = 1$ here the area under the density function has to be one. This is the area of the left triangle plus the area of the right rectangle in the plot of $f(x)$. Therefore, we should have

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{2a}{2} + 2a = 3a = 1 \Rightarrow a = \frac{1}{3}$$

2. The equation for $f(x)$ is

$$f(x) = \begin{cases} \frac{x}{6} + \frac{1}{3}, & -2 \leq x < 0 \\ \frac{1}{3}, & 0 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Here the density function is given therefore the power can be obtained from $E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$. We have

$$\begin{aligned} P_X &= \int_{-2}^{0} x^2\left(\frac{x}{6} + \frac{1}{3}\right) dx + \int_{0}^{2} x^2 \frac{1}{3}\,dx \\ &= \left[\frac{1}{24}x^4 + \frac{1}{9}x^3\right]_{-2}^{0} + \left[\frac{1}{9}x^3\right]_{0}^{2} \\ &= \frac{2}{9} + \frac{8}{9} = \frac{10}{9} \end{aligned}$$

3. Here $x_{\max} = 2$, and $N = 2^{\nu} = 32$, therefore $\nu = 5$, and

$$\begin{aligned} \text{SQNR}_{|\text{dB}} &= 4.8 + 6\nu + 10\log_{10}\frac{P_X}{x_{\max}^2} \\ &= 4.8 + 30 + 10\log_{10}\frac{10/9}{4} \\ &= 34.8 + 10\log_{10}\frac{5}{18} \\ &= 34.8 - 5.56 = 29.24 \text{ dB} \end{aligned}$$

4. $B_T = \nu W = 5 \times 12000 = 60000$ Hz.

5. Each extra bit improves SQNR by 6 dB, since we need an extra 20 dB, we need at least 4 more bits (3 more bits can improve the performance by only 18 dB), therefore the new ν will be 5+4=9 and the new bandwidth will be $B_T = 9 \times 12000 = 108000$ Hz . Compared to the previous required bandwidth of 60000 Hz, this is an increase of eighty per cent.