3) The signal $y(t)$ is periodic with period $T = T_0/\alpha$.

$$\begin{aligned}
y_n &= \frac{1}{T}\int_{\beta}^{\beta+T} y(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{\alpha}{T_0}\int_{\beta}^{\beta+\frac{T_0}{\alpha}} x(\alpha t)e^{-j2\pi\frac{n\alpha}{T_0}t}dt \\
&= \frac{1}{T_0}\int_{\beta\alpha}^{\beta\alpha+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv = x_n
\end{aligned}$$

where we used the change of variables $v = \alpha t$.

4)

$$\begin{aligned}
y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x'(t)e^{-j2\pi\frac{n}{T_0}t}dt \\
&= \frac{1}{T_0}x(t)e^{-j2\pi\frac{n}{T_0}t}\Big|_{\alpha}^{\alpha+T_0} - \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}(-j2\pi\frac{n}{T_0})e^{-j2\pi\frac{n}{T_0}t}dt \\
&= j2\pi\frac{n}{T_0}\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt = j2\pi\frac{n}{T_0}x_n
\end{aligned}$$

Problem 2.42

$$\begin{aligned}
\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)dt &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}\sum_{n=-\infty}^{\infty} x_n e^{\frac{j2\pi n}{T_0}t}\sum_{m=-\infty}^{\infty} y_m^* e^{-\frac{j2\pi m}{T_0}t}dt \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} e^{\frac{j2\pi(n-m)}{T_0}t}dt \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \delta_{mn} = \sum_{n=-\infty}^{\infty} x_n y_n^*
\end{aligned}$$

Problem 2.43

a) The signal is periodic with period T. Thus

$$\begin{aligned}
x_n &= \frac{1}{T}\int_0^T e^{-t}e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^T e^{-(j2\pi\frac{n}{T}+1)t}dt \\
&= -\frac{1}{T\left(j2\pi\frac{n}{T}+1\right)}e^{-(j2\pi\frac{n}{T}+1)t}\Big|_0^T = -\frac{1}{j2\pi n+T}\left[e^{-(j2\pi n+T)}-1\right] \\
&= \frac{1}{j2\pi n+T}[1-e^{-T}] = \frac{T-j2\pi n}{T^2+4\pi^2n^2}[1-e^{-T}]
\end{aligned}$$

If we write $x_n = \frac{a_n - jb_n}{2}$ we obtain the trigonometric Fourier series expansion coefficients as

$$a_n = \frac{2T}{T^2+4\pi^2n^2}[1-e^{-T}], \qquad b_n = \frac{4\pi n}{T^2+4\pi^2n^2}[1-e^{-T}]$$

b) The signal is periodic with period $2T$. Since the signal is odd we obtain $x_0 = 0$. For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{2T}\int_{-T}^{T} x(t)e^{-j2\pi\frac{n}{2T}t}dt = \frac{1}{2T}\int_{-T}^{T}\frac{t}{T}e^{-j2\pi\frac{n}{2T}t}dt \\
&= \frac{1}{2T^2}\int_{-T}^{T} te^{-j\pi\frac{n}{T}t}dt \\
&= \frac{1}{2T^2}\left(\frac{jT}{\pi n}te^{-j\pi\frac{n}{T}t} + \frac{T^2}{\pi^2n^2}e^{-j\pi\frac{n}{T}t}\right)\Big|_{-T}^{T} \\
&= \frac{1}{2T^2}\left[\frac{jT^2}{\pi n}e^{-j\pi n} + \frac{T^2}{\pi^2n^2}e^{-j\pi n} + \frac{jT^2}{\pi n}e^{j\pi n} - \frac{T^2}{\pi^2n^2}e^{j\pi n}\right] \\
&= \frac{j}{\pi n}(-1)^n
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \qquad b_n = (-1)^{n+1}\frac{2}{\pi n}$$

c) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{3}{2}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} + \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} \\
&= \frac{j}{2\pi n}\left[e^{-j\pi n} - e^{j\pi n} + e^{-j\pi\frac{n}{2}} - e^{-j\pi\frac{n}{2}}\right] \\
&= \frac{1}{\pi n}\sin(\pi\frac{n}{2}) = \frac{1}{2}\text{sinc}(\frac{n}{2})
\end{aligned}$$

Note that $x_n = 0$ for n even and $x_{2l+1} = \frac{1}{\pi(2l+1)}(-1)^l$. The trigonometric Fourier series expansion coefficients are:

$$a_0 = 3, \quad , a_{2l} = 0, \quad , a_{2l+1} = \frac{2}{\pi(2l+1)}(-1)^l, \quad , b_n = 0, \ \forall n$$

d) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{2}{3}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_0^T x(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^{\frac{T}{3}} \frac{3}{T}te^{-j2\pi\frac{n}{T}t}dt \\
&\quad + \frac{1}{T}\int_{\frac{T}{3}}^{\frac{2T}{3}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{2T}{3}}^{T}(-\frac{3}{T}t+3)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_0^{\frac{T}{3}} \\
&\quad - \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{\frac{2T}{3}}^{T} \\
&\quad + \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Bigg|_{\frac{T}{3}}^{\frac{2T}{3}} + \frac{3}{T}\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Bigg|_{\frac{2T}{3}}^{T} \\
&= \frac{3}{2\pi^2n^2}[\cos(\frac{2\pi n}{3}) - 1]
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = \frac{4}{3}, \quad a_n = \frac{3}{\pi^2n^2}[\cos(\frac{2\pi n}{3}) - 1], \quad b_n = 0, \ \forall n$$

e) The signal is periodic with period T. Since the signal is odd $x_0 = a_0 = 0$. For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{4}} -e^{-j2\pi\frac{n}{T}t}dt \\
&\quad + \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \frac{4}{T}te^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{4}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{4}}^{\frac{T}{4}} \\
&\quad - \frac{1}{T}\left(\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{2}}^{-\frac{T}{4}} + \frac{1}{T}\left(\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{\frac{T}{4}}^{\frac{T}{2}} \\
&= \frac{j}{\pi n}\left[(-1)^n - \frac{2\sin(\frac{\pi n}{2})}{\pi n}\right] = \frac{j}{\pi n}\left[(-1)^n - \text{sinc}(\frac{n}{2})\right]
\end{aligned}$$

For n even, $\text{sinc}(\frac{n}{2}) = 0$ and $x_n = \frac{j}{\pi n}$. The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \ \forall n, \quad b_n = \begin{cases} -\frac{1}{\pi l} & n = 2l \\ \frac{2}{\pi(2l+1)}[1 + \frac{2(-1)^l}{\pi(2l+1)}] & n = 2l+1 \end{cases}$$

f) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{3}}^{\frac{T}{3}} x(t)dt = 1$$

For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{3}}^{0}(\frac{3}{T}t+2)e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{0}^{\frac{T}{3}}(-\frac{3}{T}t+2)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{3}}^{0} \\
&\quad -\frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2n^2}e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{0}^{\frac{T}{3}} \\
&\quad +\frac{2}{T}\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Bigg|_{-\frac{T}{3}}^{0} + \frac{2}{T}\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Bigg|_{0}^{\frac{T}{3}} \\
&= \frac{3}{\pi^2n^2}\left[\frac{1}{2}-\cos(\frac{2\pi n}{3})\right] + \frac{1}{\pi n}\sin(\frac{2\pi n}{3})
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = 2, \quad a_n = 2\left[\frac{3}{\pi^2n^2}\left(\frac{1}{2}-\cos(\frac{2\pi n}{3})\right) + \frac{1}{\pi n}\sin(\frac{2\pi n}{3})\right], \quad b_n = 0, \ \forall n$$

Problem 2.44

1) $H(f) = 10\Pi(\frac{f}{4})$. The system is bandlimited with bandwidth $W = 2$. Thus at the output of the system only the frequencies in the band $[-2, \ 2]$ will be present. The gain of the filter is 10 for all f in $(-2, \ 2)$ and 5 at the edges $f = \pm 2$.

a) Since the period of the signal is $T = 1$ we obtain

$$\begin{aligned}
y(t) &= 10[\frac{a_0}{2} + a_1\cos(2\pi t) + b_1\sin(2\pi t)] \\
&\quad +5[a_2\cos(2\pi 2t) + b_2\sin(2\pi 2t)]
\end{aligned}$$

With

$$a_n = \frac{2}{1+4\pi^2n^2}[1-e^{-1}], \qquad b_n = \frac{4\pi n}{1+4\pi^2n^2}[1-e^{-1}]$$

we obtain

$$\begin{aligned}
y(t) &= (1-e^{-1})\left[20 + \frac{20}{1+4\pi^2}\cos(2\pi t) + \frac{40\pi}{1+4\pi^2}\sin(2\pi t)\right. \\
&\qquad \left. + \frac{10}{1+16\pi^2}\cos(2\pi 2t) + \frac{40\pi}{1+16\pi^2}\sin(2\pi 2t)\right]
\end{aligned}$$

b) Since the period of the signal is $2T = 2$ and $a_n = 0$, for all n, we have

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(2\pi\frac{n}{2}t)$$

The frequencies $\frac{n}{2}$ should satisfy $|\frac{n}{2}| \leq 2$ or $n \leq 4$. With $b_n = (-1)^{n+1}\frac{2}{\pi n}$ we obtain

$$\begin{aligned} y(t) &= \frac{20}{\pi}\sin(\frac{2\pi t}{2}) - \frac{20}{2\pi}\sin(2\pi t) \\ &\quad + \frac{20}{3\pi}\sin(\frac{2\pi 3t}{2}) - \frac{10}{4\pi}\sin(2\pi 2t) \end{aligned}$$

c) The period of the signal is $T = 1$ and

$$a_0 = 3, \quad , a_{2l} = 0, \quad , a_{2l+1} = \frac{2}{\pi(2l+1)}(-1)^l, \quad , b_n = 0, \ \forall n$$

Hence,

$$x(t) = \frac{3}{2} + \sum_{l=0}^{\infty} a_{2l+1}\cos(2\pi(2l+1)t)$$

At the output of the channel only the frequencies for which $2l + 1 \leq 2$ will be present so that

$$y(t) = 10\frac{3}{2} + 10\frac{2}{\pi}\cos(2\pi t)$$

d) Since $b_n = 0$ for all n, and the period of the signal is $T = 1$, we have

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nt)$$

With $a_0 = \frac{4}{3}$ and $a_n = \frac{3}{\pi^2 n^2}[\cos(\frac{2\pi n}{3}) - 1]$ we obtain

$$\begin{aligned} y(t) &= \frac{20}{3} + \frac{30}{\pi^2}(\cos(\frac{2\pi}{3}) - 1)\cos(2\pi t) \\ &\quad + \frac{15}{4\pi^2}(\cos(\frac{4\pi}{3}) - 1)\cos(2\pi 2t) \\ &= \frac{20}{3} - \frac{45}{\pi^2}\cos(2\pi t) - \frac{45}{8\pi^2}\cos(2\pi 2t) \end{aligned}$$

e) With $a_n = 0$ for all n, $T = 1$ and

$$b_n = \begin{cases} -\frac{1}{\pi l} & n = 2l \\ \frac{2}{\pi(2l+1)}[1 + \frac{2(-1)^l}{\pi(2l+1)}] & n = 2l + 1 \end{cases}$$

we obtain

$$\begin{aligned} y(t) &= 10b_1\sin(2\pi t) + 5b_2\sin(2\pi t 2t) \\ &= 10\frac{2}{\pi}(1 + \frac{2}{\pi})\sin(2\pi t) - 5\frac{1}{\pi}\sin(2\pi t 2t) \end{aligned}$$

f) Similarly with the other cases we obtain

$$\begin{aligned} y(t) &= 10 + 10 \cdot 2\left[\frac{3}{\pi^2}(\frac{1}{2} - \cos(\frac{2\pi}{3})) + \frac{1}{\pi}\sin(\frac{2\pi}{3})\right]\cos(2\pi t) \\ &\quad +5 \cdot 2\left[\frac{3}{4\pi^2}(\frac{1}{2} - \cos(\frac{4\pi}{3})) + \frac{1}{2\pi}\sin(\frac{4\pi}{3})\right]\cos(2\pi 2t) \\ &= 10 + 20\left[\frac{3}{\pi^2} + \frac{\sqrt{3}}{2\pi}\right]\cos(2\pi t) + 10\left[\frac{3}{4\pi^2} - \frac{\sqrt{3}}{4\pi}\right]\cos(2\pi 2t) \end{aligned}$$

2) In general

$$y(t) = \sum_{n=-\infty}^{\infty} x_n H(\frac{n}{T}) e^{j2\pi \frac{n}{T} t}$$

The DC component of the input signal and all frequencies higher than 4 will be cut off.

a) For this signal $T = 1$ and $x_n = \frac{1-j2\pi n}{1+4\pi^2 n^2}(1-e^{-1})$. Thus,

$$\begin{aligned} y(t) &= \frac{1-j2\pi}{1+4\pi^2}(1-e^{-1})(-j)e^{j2\pi t} + \frac{1-j2\pi 2}{1+4\pi^2 4}(1-e^{-1})(-j)e^{j2\pi 2t} \\ &\quad + \frac{1-j2\pi 3}{1+4\pi^2 9}(1-e^{-1})(-j)e^{j2\pi 3t} + \frac{1-j2\pi 4}{1+4\pi^2 16}(1-e^{-1})(-j)e^{j2\pi 4t} \\ &\quad + \frac{1+j2\pi}{1+4\pi^2}(1-e^{-1})je^{-j2\pi t} + \frac{1+j2\pi 2}{1+4\pi^2 4}(1-e^{-1})je^{-j2\pi 2t} \\ &\quad + \frac{1+j2\pi 3}{1+4\pi^2 9}(1-e^{-1})je^{-j2\pi 3t} + \frac{1+j2\pi 4}{1+4\pi^2 16}(1-e^{-1})je^{-j2\pi 4t} \\ &= (1-e^{-1})\sum_{n=1}^{4}\frac{2}{1+4\pi^2 n^2}(\sin(2\pi nt) - 2\pi n\cos(2\pi nt)) \end{aligned}$$

b) With $T = 2$ and $x_n = \frac{j}{\pi n}(-1)^n$ we obtain

$$\begin{aligned} y(t) &= \sum_{n=1}^{8}\frac{j}{\pi n}(-1)^n(-j)e^{j\pi nt} + \sum_{n=-8}^{-1}\frac{j}{\pi n}(-1)^n je^{j\pi nt} \\ &= \sum_{n=1}^{8}\frac{(-1)^n}{\pi n}e^{j\pi nt} + \sum_{n=-8}^{-1} -\frac{1}{\pi n}(-1)^n je^{j\pi nt} \end{aligned}$$

c) In this case

$$x_{2l} = 0, \qquad x_{2l+1} = \frac{1}{\pi(2l+1)}(-1)^l$$

Hence

$$\begin{aligned} y(t) &= \frac{1}{\pi}(-j)e^{j2\pi t} + \frac{1}{3\pi}(-1)(-j)e^{j2\pi 3t} \\ &\quad + \frac{1}{-\pi}(-1)je^{-j2\pi t} + \frac{1}{-3\pi}je^{-j2\pi 3t} \\ &= \frac{1}{2\pi}\sin(2\pi t) - \frac{1}{6\pi}\sin(2\pi 3t) \end{aligned}$$

d) $x_0 = \frac{2}{3}$ and $x_n = \frac{3}{2\pi n^2}(\cos(\frac{2\pi n}{3}) - 1)$. Thus

$$\begin{aligned} y(t) &= \sum_{n=1}^{4} \frac{3}{2\pi n^2}(\cos(\frac{2\pi n}{3}) - 1)(-j)e^{j2\pi nt} \\ &\quad + \sum_{n=-4}^{-1} \frac{3}{2\pi n^2}(\cos(\frac{2\pi n}{3}) - 1)je^{j2\pi nt} \end{aligned}$$

e) With $x_n = \frac{j}{\pi n}((-1)^n - \text{sinc}(\frac{n}{2}))$ we obtain

$$y(t) = \sum_{n=1}^{4} \frac{1}{\pi n}((-1)^n - \text{sinc}(\frac{n}{2})) + \sum_{n=-4}^{-1} \frac{-1}{\pi n}((-1)^n - \text{sinc}(\frac{n}{2}))$$

f) Working similarly with the other cases we obtain

$$\begin{aligned} y(t) &= \sum_{n=1}^{4} \left[\frac{3}{\pi^2 n^2}\left(\frac{1}{2} - \cos(\frac{2\pi n}{3})\right) + \frac{1}{\pi n}\sin(\frac{2\pi n}{3})\right](-j)e^{j2\pi nt} \\ &\quad + \sum_{n=-4}^{-1} \left[\frac{3}{\pi^2 n^2}\left(\frac{1}{2} - \cos(\frac{2\pi n}{3})\right) + \frac{1}{\pi n}\sin(\frac{2\pi n}{3})\right]je^{j2\pi nt} \end{aligned}$$

Problem 2.45

Using Parseval's relation (Equation 2.2.38), we see that the power in the periodic signal is given by $\sum_{n=-\infty}^{\infty} |x_n|^2$. Since the signal has finite power

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt = K < \infty$$

Thus, $\sum_{n=-\infty}^{\infty} |x_n|^2 = K < \infty$. The last implies that $|x_n| \to 0$ as $n \to \infty$. To see this write

$$\sum_{n=-\infty}^{\infty} |x_n|^2 = \sum_{n=-\infty}^{-M} |x_n|^2 + \sum_{n=-M}^{M} |x_n|^2 + \sum_{n=M}^{\infty} |x_n|^2$$

Each of the previous terms is positive and bounded by K. Assume that $|x_n|^2$ does not converge to zero as n goes to infinity and choose $\epsilon = 1$. Then there exists a subsequence of x_n, x_{n_k}, such that

$$|x_{n_k}| > \epsilon = 1, \qquad \text{for } n_k > N \geq M$$

Then

$$\sum_{n=M}^{\infty} |x_n|^2 \geq \sum_{n=N}^{\infty} |x_n|^2 \geq \sum_{n_k} |x_{n_k}|^2 = \infty$$

This contradicts our assumption that $\sum_{n=M}^{\infty} |x_n|^2$ is finite. Thus $|x_n|$, and consequently x_n, should converge to zero as $n \to \infty$.

Problem 2.46

1) Using the Fourier transform pair

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + (2\pi f)^2} = \frac{2\alpha}{4\pi^2} \frac{1}{\frac{\alpha^2}{4\pi^2} + f^2}$$

and the duality property of the Fourier transform: $X(f) = \mathcal{F}[x(t)] \Rightarrow x(-f) = \mathcal{F}[X(t)]$ we obtain

$$\left(\frac{2\alpha}{4\pi^2}\right) \mathcal{F}\left[\frac{1}{\frac{\alpha^2}{4\pi^2} + t^2}\right] = e^{-\alpha|f|}$$

With $\alpha = 2\pi$ we get the desired result

$$\mathcal{F}\left[\frac{1}{1+t^2}\right] = \pi e^{-2\pi|f|}$$

2)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3) + \Pi(t+3)] \\
&= \text{sinc}(f)e^{-j2\pi f3} + \text{sinc}(f)e^{j2\pi f3} \\
&= 2\text{sinc}(f)\cos(2\pi 3f)
\end{aligned}$$

3)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[\Lambda(2t+3) + \Lambda(3t-2)] \\
&= \mathcal{F}[\Lambda(2(t+\frac{3}{2})) + \Lambda(3(t-\frac{2}{3})] \\
&= \frac{1}{2}\text{sinc}^2(\frac{f}{2})e^{j\pi f3} + \frac{1}{3}\text{sinc}^2(\frac{f}{3})e^{-j2\pi f\frac{2}{3}}
\end{aligned}$$

4) $\mathcal{F}[\Pi(t/4)] = 4\text{sinc}(4f)$, hence $\mathcal{F}[4\Pi(t/4)] = 16\text{sinc}(4f)$. Using modulation property of FT we have $\mathcal{F}[4\Pi(t/4)\cos(2\pi f_0 t)] = 8\text{sinc}(4(f-f_0)) + 8\text{sinc}(4(f+f_0))$.

5) We use a combination of scaling, time shift, and modulation properties to obtain the result. $\mathcal{F}\left[4\Pi\left(\frac{t-2}{4}\right)\right] = 16e^{-j4\pi f}\text{sinc}(4f)$ and

$$\mathcal{F}\left[4\Pi\left(\frac{t-2}{4}\right)\cos(2\pi f_0 t)\right] = 8e^{-j4\pi(f-f_0)}\text{sinc}(4(f-f_0)) + 8e^{-j4\pi(f+f_0)}\text{sinc}(4(f+f_0))$$

6) $T(f) = \mathcal{F}[\text{sinc}^3(t)] = \mathcal{F}[\text{sinc}^2(t)\text{sinc}(t)] = \Lambda(f) \star \Pi(f)$. But

$$\Pi(f) \star \Lambda(f) = \int_{-\infty}^{\infty} \Pi(\theta)\Lambda(f-\theta)d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(f-\theta)d\theta = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \Lambda(v)dv$$

For $\quad f \le -\frac{3}{2} \Longrightarrow T(f) = 0$

For $\quad -\frac{3}{2} < f \le -\frac{1}{2} \Longrightarrow T(f) = \int_{-1}^{f+\frac{1}{2}} (v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{f+\frac{1}{2}} = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8}$

For $\quad -\frac{1}{2} < f \le \frac{1}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{0} (v+1)dv + \int_{0}^{f+\frac{1}{2}} (-v+1)dv$

$$= (\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{0} + (-\frac{1}{2}v^2 + v)\Big|_{0}^{f+\frac{1}{2}} = -f^2 + \frac{3}{4}$$

For $\quad \frac{1}{2} < f \le \frac{3}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{1} (-v+1)dv = (-\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{1} = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8}$

For $\quad \frac{3}{2} < f \Longrightarrow T(f) = 0$

Thus,

$$T(f) = \begin{cases} 0 & f \le -\frac{3}{2} \\ \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} & -\frac{3}{2} < f \le -\frac{1}{2} \\ -f^2 + \frac{3}{4} & -\frac{1}{2} < f \le \frac{1}{2} \\ \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} & \frac{1}{2} < f \le \frac{3}{2} \\ 0 & \frac{3}{2} < f \end{cases}$$

7)

$$\mathcal{F}[t\,\text{sinc}(t)] = \frac{1}{\pi}\mathcal{F}[\sin(\pi t)] = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2}) - \delta(f-\frac{1}{2})\right]$$

The same result is obtain if we recognize that multiplication by t results in differentiation in the frequency domain. Thus

$$\mathcal{F}[t\,\text{sinc}] = \frac{j}{2\pi}\frac{d}{df}\Pi(f) = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2}) - \delta(f-\frac{1}{2})\right]$$

8)

$$\begin{aligned} \mathcal{F}[t\cos(2\pi f_0 t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right) \\ &= \frac{j}{4\pi}\left(\delta'(f-f_0) + \delta'(f+f_0)\right) \end{aligned}$$

9)

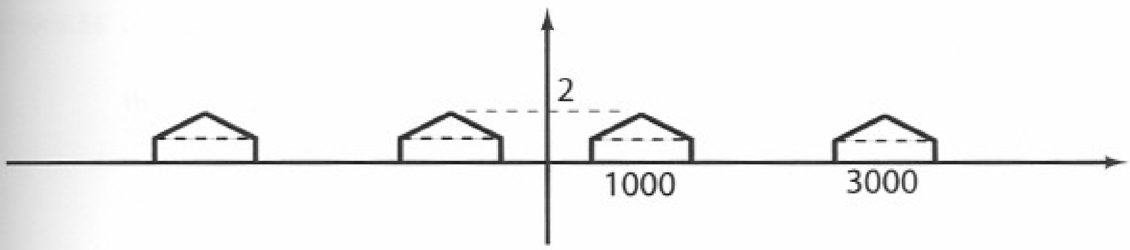
$$\mathcal{F}[e^{-\alpha|t|}\cos(\beta t)] = \frac{1}{2}\left[\frac{2\alpha}{\alpha^2 + (2\pi(f - \frac{\beta}{2\pi}))^2} + \frac{2\alpha}{\alpha^2 + (2\pi(f + \frac{\beta}{2\pi}))^2}\right]$$

10)

$$\begin{aligned}\mathcal{F}[te^{-\alpha|t|}\cos(\beta t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{\alpha}{\alpha^2 + (2\pi(f - \frac{\beta}{2\pi}))^2} + \frac{\alpha}{\alpha^2 + (2\pi(f + \frac{\beta}{2\pi}))^2}\right) \\ &= -j\left[\frac{2\alpha\pi(f - \frac{\beta}{2\pi})}{\left(\alpha^2 + (2\pi(f - \frac{\beta}{2\pi}))^2\right)^2} + \frac{2\alpha\pi(f + \frac{\beta}{2\pi})}{\left(\alpha^2 + (2\pi(f + \frac{\beta}{2\pi}))^2\right)^2}\right]\end{aligned}$$

Problem 2.47

$x_1(t) = -x(t) + x(t)\cos(2000\pi t) + x(t)\,(1 + \cos(6000\pi t))$ or $x_1(t) = x(t)\cos(2000\pi t) + x(t)\cos(6000\pi t)$. Using modulation property, we have $X_1(f) = \frac{1}{2}X(f - 1000) + \frac{1}{2}X(f + 1000) + \frac{1}{2}X(f - 3000) + \frac{1}{2}X(f + 3000)$. The plot is given below:



Problem 2.48

$$\begin{aligned}\mathcal{F}[\frac{1}{2}(\delta(t + \frac{1}{2}) + \delta(t - \frac{1}{2}))] &= \int_{-\infty}^{\infty}\frac{1}{2}(\delta(t + \frac{1}{2}) + \delta(t - \frac{1}{2}))e^{-j2\pi ft}dt \\ &= \frac{1}{2}(e^{-j\pi f} + e^{-j\pi f}) = \cos(\pi f)\end{aligned}$$

Using the duality property of the Fourier transform:

$$X(f) = \mathcal{F}[x(t)] \Longrightarrow x(f) = \mathcal{F}[X(-t)]$$

we obtain

$$\mathcal{F}[\cos(-\pi t)] = \mathcal{F}[\cos(\pi t)] = \frac{1}{2}(\delta(f+\frac{1}{2}) + \delta(f-\frac{1}{2}))$$

Note that $\sin(\pi t) = \cos(\pi t + \frac{\pi}{2})$. Thus

$$\begin{aligned}
\mathcal{F}[\sin(\pi t)] &= \mathcal{F}[\cos(\pi(t+\frac{1}{2}))] = \frac{1}{2}(\delta(f+\frac{1}{2}) + \delta(f-\frac{1}{2}))e^{j\pi f} \\
&= \frac{1}{2}e^{j\pi\frac{1}{2}}\delta(f+\frac{1}{2}) + \frac{1}{2}e^{-j\pi\frac{1}{2}}\delta(f-\frac{1}{2}) \\
&= \frac{j}{2}\delta(f+\frac{1}{2}) - \frac{j}{2}\delta(f-\frac{1}{2})
\end{aligned}$$

Problem 2.49

a) We can write $x(t)$ as $x(t) = 2\Pi(\frac{t}{4}) - 2\Lambda(\frac{t}{2})$. Then

$$\mathcal{F}[x(t)] = \mathcal{F}[2\Pi(\frac{t}{4})] - \mathcal{F}[2\Lambda(\frac{t}{2})] = 8\text{sinc}(4f) - 4\text{sinc}^2(2f)$$

b)

$$x(t) = 2\Pi(\frac{t}{4}) - \Lambda(t) \Longrightarrow \mathcal{F}[x(t)] = 8\text{sinc}(4f) - \text{sinc}^2(f)$$

c)

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{-1}^{0}(t+1)e^{-j2\pi ft}dt + \int_{0}^{1}(t-1)e^{-j2\pi ft}dt \\
&= \left(\frac{j}{2\pi f}t + \frac{1}{4\pi^2 f^2}\right)e^{-j2\pi ft}\Big|_{-1}^{0} + \frac{j}{2\pi f}e^{-j2\pi ft}\Big|_{-1}^{0} \\
&\quad + \left(\frac{j}{2\pi f}t + \frac{1}{4\pi^2 f^2}\right)e^{-j2\pi ft}\Big|_{0}^{1} - \frac{j}{2\pi f}e^{-j2\pi ft}\Big|_{0}^{1} \\
&= \frac{j}{\pi f}(1 - \sin(\pi f))
\end{aligned}$$

d) We can write $x(t)$ as $x(t) = \Lambda(t+1) - \Lambda(t-1)$. Thus

$$X(f) = \text{sinc}^2(f)e^{j2\pi f} - \text{sinc}^2(f)e^{-j2\pi f} = 2j\text{sinc}^2(f)\sin(2\pi f)$$

e) We can write $x(t)$ as $x(t) = \Lambda(t+1) + \Lambda(t) + \Lambda(t-1)$. Hence,

$$X(f) = \text{sinc}^2(f)(1 + e^{j2\pi f} + e^{-j2\pi f}) = \text{sinc}^2(f)(1 + 2\cos(2\pi f)$$

f) We can write $x(t)$ as

$$x(t) = \left[\Pi\left(2f_0(t - \frac{1}{4f_0})\right) - \Pi\left(2f_0(t - \frac{1}{4f_0})\right)\right]\sin(2\pi f_0 t)$$

Then

$$\begin{aligned} X(f) &= \left[\frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0}\right)e^{-j2\pi\frac{1}{4f_0}f} - \frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0})\right)e^{j2\pi\frac{1}{4f_0}f}\right] \\ &\quad \star\frac{j}{2}(\delta(f+f_0) - \delta(f+f_0)) \\ &= \frac{1}{2f_0}\text{sinc}\left(\frac{f+f_0}{2f_0}\right)\sin\left(\pi\frac{f+f_0}{2f_0}\right) - \frac{1}{2f_0}\text{sinc}\left(\frac{f-f_0}{2f_0}\right)\sin\left(\pi\frac{f-f_0}{2f_0}\right) \end{aligned}$$

Problem 2.50

(Convolution theorem:)

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)]\mathcal{F}[y(t)] = X(f)Y(f)$$

Thus

$$\begin{aligned} \text{sinc}(t) \star \text{sinc}(t) &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t) \star \text{sinc}(t)]] \\ &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t)] \cdot \mathcal{F}[\text{sinc}(t)]] \\ &= \mathcal{F}^{-1}[\Pi(f)\Pi(f)] = \mathcal{F}^{-1}[\Pi(f)] \\ &= \text{sinc}(t) \end{aligned}$$

Problem 2.51

$$\begin{aligned} \mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(\theta)e^{j2\pi\theta t}d\theta\right)y(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} X(\theta)\left(\int_{-\infty}^{\infty} y(t)e^{-j2\pi(f-\theta)t}dt\right)d\theta \\ &= \int_{-\infty}^{\infty} X(\theta)Y(f-\theta)d\theta = X(f) \star Y(f) \end{aligned}$$

Problem 2.52

1) Clearly

$$\begin{aligned} x_1(t+kT_0) &= \sum_{n=-\infty}^{\infty} x(t+kT_0-nT_0) = \sum_{n=-\infty}^{\infty} x(t-(n-k)T_0) \\ &= \sum_{m=-\infty}^{\infty} x(t-mT_0) = x_1(t) \end{aligned}$$

Thus

$$y(t) = \begin{cases} \frac{1}{\alpha+\beta} e^{\alpha t} & t \leq 0 \\ -\frac{2\alpha e^{-\beta t}}{\beta^2-\alpha^2} + \frac{e^{-\alpha t}}{\beta-\alpha} & t > 0 \end{cases}$$

In the case of $\alpha = \beta$

$$\begin{aligned} \text{If } t < 0 \Rightarrow y(t) &= e^{-\alpha t} \int_{-\infty}^{t} e^{2\alpha\tau} d\tau = \frac{1}{2\alpha} e^{\alpha t} \\ \text{If } t < 0 \Rightarrow y(t) &= \int_{-\infty}^{0} e^{-\alpha t} e^{2\alpha\tau} d\tau + \int_{0}^{t} e^{-\alpha t} d\tau \\ &= \frac{e^{-\alpha t}}{2\alpha} e^{2\alpha\tau}\Big|_{-\infty}^{0} + t e^{-\alpha t} \\ &= [\frac{1}{2\alpha} + t] e^{-\alpha t} \end{aligned}$$

6) Using the convolution theorem we obtain

$$Y(f) = \Pi(f)\Lambda(f) = \begin{cases} 0 & \frac{1}{2} < |f| \\ f+1 & -\frac{1}{2} < f \leq 0 \\ -f+1 & 0 \leq f < \frac{1}{2} \end{cases}$$

Thus

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}[Y(f)] = \int_{-\frac{1}{2}}^{\frac{1}{2}} Y(f) e^{j2\pi f t} df \\ &= \int_{-\frac{1}{2}}^{0} (f+1) e^{j2\pi f t} df + \int_{0}^{\frac{1}{2}} (-f+1) e^{j2\pi f t} df \\ &= \left(\frac{1}{j2\pi t} f e^{j2\pi f t} + \frac{1}{4\pi^2 t^2} e^{j2\pi f t}\right)\Big|_{-\frac{1}{2}}^{0} + \frac{1}{j2\pi t} e^{j2\pi f t}\Big|_{-\frac{1}{2}}^{0} \\ &\quad - \left(\frac{1}{j2\pi t} f e^{j2\pi f t} + \frac{1}{4\pi^2 t^2} e^{j2\pi f t}\right)\Big|_{0}^{\frac{1}{2}} + \frac{1}{j2\pi t} e^{j2\pi f t}\Big|_{0}^{\frac{1}{2}} \\ &= \frac{1}{2\pi^2 t^2}[1 - \cos(\pi t)] + \frac{1}{2\pi t} \sin(\pi t) \end{aligned}$$

Problem 2.60

Let the response of the LTI system be $h(t)$ with Fourier transform $H(f)$. Then, from the convolution theorem we obtain

$$Y(f) = H(f)X(f) \Longrightarrow \Lambda(f) = \Pi(f)H(f)$$

However, this relation cannot hold since $\Pi(f) = 0$ for $\frac{1}{2} < |f|$ whereas $\Lambda(f) \neq 0$ for $1 < |f| \leq 1/2$.

Problem 2.61

1) No. The input $\Pi(t)$ has a spectrum with zeros at frequencies $f = k, (k \neq 0, k \in \mathcal{Z})$ and the information about the spectrum of the system at those frequencies will not be present at the output. The spectrum of the signal $\cos(2\pi t)$ consists of two impulses at $f = \pm 1$ but we do not know the response of the system at these frequencies.

2)

$$\begin{aligned} h_1(t) \star \Pi(t) &= \Pi(t) \star \Pi(t) = \Lambda(t) \\ h_2(t) \star \Pi(t) &= (\Pi(t) + \cos(2\pi t)) \star \Pi(t) \\ &= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\text{sinc}^2(f) + \delta(f+1)\text{sinc}^2(f)\right] \\ &= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\text{sinc}^2(1) + \delta(f+1)\text{sinc}^2(-1)\right] \\ &= \Lambda(t) \end{aligned}$$

Thus both signals are candidates for the impulse response of the system.

3) $\mathcal{F}[u_{-1}(t)] = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$. Thus the system has a nonzero spectrum for every f and all the frequencies of the system will be excited by this input. $\mathcal{F}[e^{-at}u_{-1}(t)] = \frac{1}{a+j2\pi f}$. Again the spectrum is nonzero for all f and the response to this signal uniquely determines the system. In general the spectrum of the input must not vanish at any frequency. In this case the influence of the system will be present at the output for every frequency.

Problem 2.62

$$\begin{aligned} \mathcal{F}[A\sin\widehat{(2\pi f_0 t + \theta)}] &= -j\text{sgn}(f)A\left[-\frac{1}{2j}\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}} + \frac{1}{2j}\delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\ &= \frac{A}{2}\left[\text{sgn}(-f_0)\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}} - \text{sgn}(-f_0)\delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\ &= -\frac{A}{2}\left[\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}} + \delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\ &= -A\mathcal{F}[\cos(2\pi f_0 t + \theta)] \end{aligned}$$

Thus, $A\sin\widehat{(2\pi f_0 t + \theta)} = -A\cos(2\pi f_0 t + \theta)$

Problem 2.63

Taking the Fourier transform of $\widehat{e^{j2\pi f_0 t}}$ we obtain

$$\mathcal{F}[\widehat{e^{j2\pi f_0 t}}] = -j\text{sgn}(f)\delta(f-f_0) = -j\text{sgn}(f_0)\delta(f-f_0)$$

Thus,

$$\widehat{e^{j2\pi f_0 t}} = \mathcal{F}^{-1}[-j\text{sgn}(f_0)\delta(f-f_0)] = -j\text{sgn}(f_0)e^{j2\pi f_0 t}$$