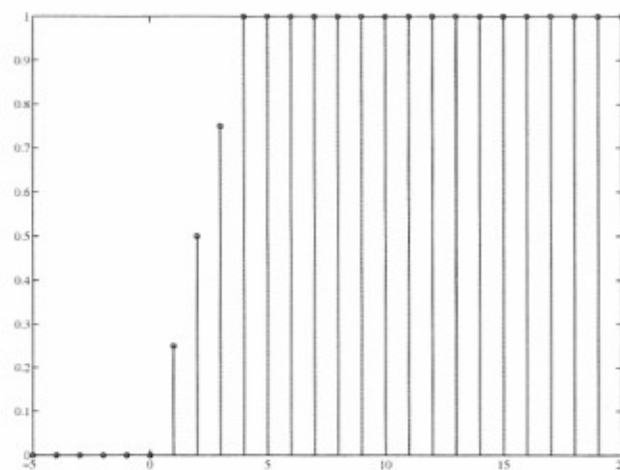


3.  $x[n] = \frac{n}{4}u_{-1}(n/4) - (\frac{n}{4} - 1)u_{-1}(n/4 - 1)$ . For  $n < 0$ ,  $x[n] = 0$ , for  $0 \leq n \leq 3$ ,  $x[n] = \frac{n}{4}$  and for  $n \geq 4$ ,  $x[n] = \frac{n}{4} - \frac{n}{4} + 1 = 1$ .




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### Problem 2.3

$x_1[n] = 1$  and  $x_2[n] = \cos(2\pi n) = 1$ , for all  $n$ . This shows that two signals can be different but their sampled versions be the same.

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### Problem 2.4

Let  $x_1[n]$  and  $x_2[n]$  be two periodic signals with periods  $N_1$  and  $N_2$ , respectively, and let  $N = \text{LCM}(N_1, N_2)$ , and define  $x[n] = x_1[n] + x_2[n]$ . Then obviously  $x_1[n + N] = x_1[n]$  and  $x_2[n + N] = x_2[n]$ , and hence  $x[n] = x[n + N]$ , i.e.,  $x[n]$  is periodic with period  $N$ .

For continuous-time signals  $x_1(t)$  and  $x_2(t)$  with periods  $T_1$  and  $T_2$  respectively, in general we cannot find a  $T$  such that  $T = k_1 T_1 = k_2 T_2$  for integers  $k_1$  and  $k_2$ . This is obvious for instance if  $T_1 = 1$  and  $T_2 = \pi$ . The necessary and sufficient condition for the sum to be periodic is that  $\frac{T_1}{T_2}$  be a rational number.

### Problem 2.5

Using the result of problem 2.4 we have:

1. The frequencies are 2000 and 5500, their ratio (and therefore the ratio of the periods) is rational, hence the sum is periodic.
2. The frequencies are 2000 and  $\frac{5500}{\pi}$ . Their ratio is not rational, hence the sum is not periodic.
3. The sum of two periodic discrete-time signal is periodic.
4. The first signal is periodic but  $\cos[11000n]$  is *not* periodic, since there is no  $N$  such that  $\cos[11000(n+N)] = \cos(11000n)$  for all  $n$ . Therefore the sum cannot be periodic.

### Problem 2.6

1)

$$x_1(t) = \begin{cases} e^{-t} & t > 0 \\ -e^t & t < 0 \\ 0 & t = 0 \end{cases} \implies x_1(-t) = \begin{cases} -e^{-t} & t > 0 \\ e^t & t < 0 \\ 0 & t = 0 \end{cases} = -x_1(t)$$

Thus,  $x_1(t)$  is an odd signal

2)  $x_2(t) = \cos(120\pi t + \frac{\pi}{3})$  is neither even nor odd. We have  $\cos(120\pi t + \frac{\pi}{3}) = \cos(\frac{\pi}{3})\cos(120\pi t) - \sin(\frac{\pi}{3})\sin(120\pi t)$ . Therefore  $x_{2e}(t) = \cos(\frac{\pi}{3})\cos(120\pi t)$  and  $x_{2o}(t) = -\sin(\frac{\pi}{3})\sin(120\pi t)$ . (Note: This part can also be considered as a special case of part 7 of this problem)

3)

$$x_3(t) = e^{-|t|} \implies x_3(-t) = e^{-|(-t)|} = e^{-|t|} = x_3(t)$$

Hence, the signal  $x_3(t)$  is even.

4)

$$x_4(t) = \begin{cases} \frac{t}{|t|} & t \neq 0 \\ 0 & t = 0 \end{cases} \implies x_4(-t) = \begin{cases} \frac{-t}{|t|} & t \neq 0 \\ 0 & t = 0 \end{cases} = -x_4(t)$$

Thus, the signal  $x_4(t)$  is odd.

5)

$$x_5(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \implies x_5(-t) = \begin{cases} 0 & t \geq 0 \\ -t & t < 0 \end{cases}$$

The signal  $x_5(t)$  is neither even nor odd. The even part of the signal is

$$x_{5,e}(t) = \frac{x_5(t) + x_5(-t)}{2} = \begin{cases} \frac{t}{2} & t \geq 0 \\ \frac{-t}{2} & t < 0 \end{cases} = \frac{|t|}{2}$$

The odd part is

$$x_{5,o}(t) = \frac{x_5(t) - x_5(-t)}{2} = \begin{cases} \frac{t}{2} & t \geq 0 \\ \frac{t}{2} & t < 0 \end{cases} = \frac{t}{2}$$

6)

$$x_6(t) = \sin t + \cos t \implies x_6(-t) = -\sin t + \cos t$$

Clearly  $x_6(-t) \neq x_6(t)$  for every  $t$  since otherwise  $2 \sin t = 0 \forall t$ . Similarly  $x_6(-t) \neq -x_6(t)$  for every  $t$  since otherwise  $2 \cos t = 0 \forall t$ . Thus  $x_6(t)$  is neither even or odd. The even and the odd parts of  $x_6(t)$  are given by

$$\begin{aligned} x_{6,e}(t) &= \frac{x_6(t) + x_6(-t)}{2} = \cos t \\ x_{6,o}(t) &= \frac{x_6(t) - x_6(-t)}{2} = \sin t \end{aligned}$$

7)

$$x_7(t) = x_1(t) - x_2(t) \implies x_7(-t) = x_1(-t) - x_2(-t) = x_1(t) + x_2(t)$$

Clearly  $x_7(-t) \neq x_7(t)$  since otherwise  $x_2(t) = 0 \forall t$ . Similarly  $x_7(-t) \neq -x_7(t)$  since otherwise  $x_1(t) = 0 \forall t$ . The even and the odd parts of  $x_7(t)$  are given by

$$\begin{aligned} x_{7,e}(t) &= \frac{x_7(t) + x_7(-t)}{2} = x_1(t) \\ x_{7,o}(t) &= \frac{x_7(t) - x_7(-t)}{2} = -x_2(t) \end{aligned}$$

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### Problem 2.7

For the first two questions we will need the integral  $I = \int e^{ax} \cos^2 x dx$ .

$$\begin{aligned} I &= \frac{1}{a} \int \cos^2 x de^{ax} = \frac{1}{a} e^{ax} \cos^2 x + \frac{1}{a} \int e^{ax} \sin 2x dx \\ &= \frac{1}{a} e^{ax} \cos^2 x + \frac{1}{a^2} \int \sin 2x de^{ax} \\ &= \frac{1}{a} e^{ax} \cos^2 x + \frac{1}{a^2} e^{ax} \sin 2x - \frac{2}{a^2} \int e^{ax} \cos 2x dx \\ &= \frac{1}{a} e^{ax} \cos^2 x + \frac{1}{a^2} e^{ax} \sin 2x - \frac{2}{a^2} \int e^{ax} (2 \cos^2 x - 1) dx \\ &= \frac{1}{a} e^{ax} \cos^2 x + \frac{1}{a^2} e^{ax} \sin 2x - \frac{2}{a^2} \int e^{ax} dx - \frac{4}{a^2} I \end{aligned}$$

Thus,

$$I = \frac{1}{4 + a^2} \left[ (a \cos^2 x + \sin 2x) + \frac{2}{a} \right] e^{ax}$$

2.  $x_2(t + 1) = \sum_{n=-\infty}^{\infty} \Lambda(t - n + 1) = \sum_{n=-\infty}^{\infty} \Lambda(t - (n - 1)) = \sum_{n=-\infty}^{\infty} \Lambda(t - n) = x_2(t)$ . Hence  $x_2(t)$  is periodic with period 1.
3. This is the sum of two periodic signals with periods  $2\pi$  and 1. Since the ratio of the two periods is not rational the sum is not periodic (by the result of problem 2.4)
4.  $\sin[n]$  is not periodic. There is no integer  $N$  such that  $\sin[n + N] = \sin[n]$  for all  $n$ .
5.  $x_5(t + T) = \sum_{n=-\infty}^{\infty} y(t - nT + T) = \sum_{n=-\infty}^{\infty} y(t - (n - 1)T) = \sum_{n=-\infty}^{\infty} y(t - nT) = x_5(t)$ . Therefore  $x_5(t)$  is periodic with period  $T$ .

### Problem 2.9

1)

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 |e^{j(2\pi f_0 t + \theta)}|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} A^2 T = A^2$$

Thus  $x(t) = Ae^{j(2\pi f_0 t + \theta)}$  is a power-type signal and its power content is  $A^2$ .

2)

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \cos^2(2\pi f_0 t + \theta) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A^2}{2} dt + \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A^2}{2} \cos(4\pi f_0 t + 2\theta) dt$$

As  $T \rightarrow \infty$ , there will be no contribution by the second integral. Thus the signal is a power-type signal and its power content is  $\frac{A^2}{2}$ .

3)

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} u_{-1}^2(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \frac{T}{2} = \frac{1}{2}$$

Thus the unit step signal is a power-type signal and its power content is  $1/2$

4)

$$\begin{aligned} E_x &= \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} x^2(t) dt = \lim_{T \rightarrow \infty} \int_0^{T/2} K^2 t^{-1/2} dt = \lim_{T \rightarrow \infty} 2K^2 t^{1/2} \Big|_0^{T/2} \\ &= \lim_{T \rightarrow \infty} \sqrt{2} K^2 T^{1/2} = \infty \end{aligned}$$

Thus the signal is not an energy-type signal.

$$\begin{aligned} P_x &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} K^2 t^{-1/2} dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} 2K^2 t^{1/2} \Big|_0^{T/2} = \lim_{T \rightarrow \infty} \frac{1}{T} 2K^2 (T/2)^{1/2} = \lim_{T \rightarrow \infty} \sqrt{2} K^2 T^{-1/2} = 0 \end{aligned}$$

Since  $P_x$  is not bounded away from zero it follows by definition that the signal is not of the power-type (recall that power-type signals should satisfy  $0 < P_x < \infty$ ).

**Problem 2.10**

$$\Lambda(t) = \begin{cases} t+1, & -1 \leq t \leq 0 \\ -t+1, & 0 \leq t \leq 1 \\ 0, & \text{o.w.} \end{cases} \quad u_{-1}(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$

Thus, the signal  $x(t) = \Lambda(t)u_{-1}(t)$  is given by

$$x(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ -t+1 & 0 \leq t \leq 1 \\ 0 & t \geq 1 \end{cases} \implies x(-t) = \begin{cases} 0 & t \leq -1 \\ t+1 & -1 \leq t < 0 \\ 1/2 & t = 0 \\ 0 & t > 0 \end{cases}$$

The even and the odd part of  $x(t)$  are given by

$$x_e(t) = \frac{x(t) + x(-t)}{2} = \frac{1}{2}\Lambda(t)$$

$$x_o(t) = \frac{x(t) - x(-t)}{2} = \begin{cases} 0 & t \leq -1 \\ \frac{-t-1}{2} & -1 \leq t < 0 \\ 0 & t = 0 \\ \frac{-t+1}{2} & 0 < t \leq 1 \\ 0 & 1 \leq t \end{cases}$$

**Problem 2.11**

1) Suppose that

$$x(t) = x_e^1(t) + x_o^1(t) = x_e^2(t) + x_o^2(t)$$

with  $x_e^1(t), x_o^2(t)$  even signals and  $x_o^1(t), x_e^2(t)$  odd signals. Then,  $x(-t) = x_e^1(t) - x_o^1(t)$  so that

$$\begin{aligned} x_e^1(t) &= \frac{x(t) + x(-t)}{2} \\ &= \frac{x_e^2(t) + x_o^2(t) + x_e^2(-t) + x_o^2(-t)}{2} \\ &= \frac{2x_e^2(t) + x_o^2(t) - x_o^2(t)}{2} = x_e^2(t) \end{aligned}$$

Thus  $x_e^1(t) = x_e^2(t)$  and  $x_o^1(t) = x(t) - x_e^1(t) = x(t) - x_e^2(t) = x_o^2(t)$

Thus, if  $n = 2l$  (even)

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(-t)dt = \int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t)dt$$

and the function  $\delta^{(n)}(t)$  is even. If  $n = 2l + 1$  (odd), then  $(-1)^n = -1$  and

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(-t)dt = - \int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t)dt$$

from which we conclude that  $\delta^{(n)}(t)$  is odd.

---

### Problem 2.15

$$x(t) \star \delta^{(n)}(t) = \int_{-\infty}^{\infty} x(\tau)\delta^{(n)}(t - \tau) d\tau$$

The signal  $\delta^{(n)}(t)$  is even if  $n$  is even and odd if  $n$  is odd. Consider first the case that  $n = 2l$ . Then,

$$x(t) \star \delta^{(2l)}(t) = \int_{-\infty}^{\infty} x(\tau)\delta^{(2l)}(\tau - t) d\tau = (-1)^{2l} \frac{d^{2l}}{d\tau^{2l}} x(\tau) \Big|_{\tau=t} = \frac{d^{2l}}{dt^{2l}} x(t)$$

If  $n$  is odd then,

$$\begin{aligned} x(t) \star \delta^{(2l+1)}(t) &= \int_{-\infty}^{\infty} x(\tau)(-1)\delta^{(2l+1)}(\tau - t) d\tau = (-1)(-1)^{2l+1} \frac{d^{2l+1}}{d\tau^{2l+1}} x(\tau) \Big|_{\tau=t} \\ &= \frac{d^{2l+1}}{dt^{2l+1}} x(t) \end{aligned}$$

In both cases

$$x(t) \star \delta^{(n)}(t) = \frac{d^n}{dt^n} x(t)$$

The convolution of  $x(t)$  with  $u_{-1}(t)$  is

$$x(t) \star u_{-1}(t) = \int_{-\infty}^{\infty} x(\tau)u_{-1}(t - \tau)d\tau$$

But  $u_{-1}(t - \tau) = 0$  for  $\tau > t$  so that

$$x(t) \star u_{-1}(t) = \int_{-\infty}^t x(\tau)d\tau$$

---

### Problem 2.16

- 1) Nonlinear, since the response to  $x(t) = 0$  is not  $y(t) = 0$  (this is a necessary condition for linearity of a system, see also problem 2.21).
- 2) Nonlinear, if we multiply the input by constant  $-1$ , the output does not change. In a linear system the output should be scaled by  $-1$ .

3) Linear, the output to any input zero, therefore for the input  $\alpha x_1(t) + \beta x_2(t)$  the output is zero which can be considered as  $\alpha y_1(t) + \beta y_2(t) = \alpha \times 0 + \beta \times 0 = 0$ . This is a linear combination of the corresponding outputs to  $x_1(t)$  and  $x_2(t)$ .

4) Nonlinear, the output to  $x(t) = 0$  is not zero.

5) Nonlinear. The system is not homogeneous for if  $\alpha < 0$  and  $x(t) > 0$  then  $y(t) = T[\alpha x(t)] = 0$  whereas  $z(t) = \alpha T[x(t)] = \alpha$ .

6) Nonlinear. The system is not homogeneous for if  $x(t) \neq 0$  then

$$y(t) = T[\alpha x(t)] = \frac{\alpha x(t)}{|\alpha||x(t)|} = \text{sgn}(\alpha) \frac{x(t)}{|x(t)|}$$

whereas

$$z(t) = \alpha T[x(t)] = \alpha \frac{x(t)}{|x(t)|}$$

7) Nonlinear. The system is not homogeneous for if  $\alpha < 0$  then  $y(t) = T[\alpha x(t)] = |\alpha||x(t)|$  whereas  $z(t) = \alpha T[x(t)] = \alpha|x(t)|$ . The system is not additive either since  $|x_1(t) + x_2(t)| \neq |x_1(t)| + |x_2(t)|$ .

8) Linear. For if  $x(t) = \alpha x_1(t) + \beta x_2(t)$  then

$$\begin{aligned} T[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))e^{-t} \\ &= \alpha x_1(t)e^{-t} + \beta x_2(t)e^{-t} = \alpha T[x_1(t)] + \beta T[x_2(t)] \end{aligned}$$

9) Linear. For if  $x(t) = \alpha x_1(t) + \beta x_2(t)$  then

$$\begin{aligned} T[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))u(t) \\ &= \alpha x_1(t)u(t) + \beta x_2(t)u(t) = \alpha T[x_1(t)] + \beta T[x_2(t)] \end{aligned}$$

10) Linear.

$$\begin{aligned} y(t) &= (\alpha x_1(t) + \beta x_2(t))\delta(t) = (\alpha x_1(0) + \beta x_2(0))\delta(t) \\ &= \alpha x_1(0)\delta(t) + \beta x_2(0)\delta(t) = \alpha x_1(t)\delta(t) + \beta x_2(t)\delta(t) \end{aligned}$$

11) Linear.

$$\begin{aligned} y(t) &= (\alpha x_1(t) + \beta x_2(t)) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} (\alpha x_1(nT) + \beta x_2(nT))\delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} \alpha x_1(nT)\delta(t - nT) + \sum_{n=-\infty}^{\infty} \beta x_2(nT)\delta(t - nT) \\ &= \alpha x_1(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) + \beta x_2(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \end{aligned}$$

12) Linear. For if  $x(t) = \alpha x_1(t) + \beta x_2(t)$  then

$$\begin{aligned} T[\alpha x_1(t) + \beta x_2(t)] &= \begin{cases} \frac{d}{dt}(\alpha x_1(t) + \beta x_2(t)) & t > 0 \\ (\alpha x_1(t) + \beta x_2(t)) & t < 0 \\ 0 & t = 0 \end{cases} \\ &= \begin{cases} \alpha \frac{d}{dt}x_1(t) + \beta \frac{d}{dt}x_2(t) & t > 0 \\ \alpha x_1(t) + \beta x_2(t) & t < 0 \\ 0 & t = 0 \end{cases} \\ &= \alpha T[x_1(t)] + \beta T[x_2(t)] \end{aligned}$$

13) Linear. We can write the output of this feedback system as

$$y(t) = x(t) + y(t-1) = \sum_{n=0}^{\infty} x(t-n)$$

Then for  $x(t) = \alpha x_1(t) + \beta x_2(t)$

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} (\alpha x_1(t-n) + \beta x_2(t-n)) \\ &= \alpha \sum_{n=0}^{\infty} x_1(t-n) + \beta \sum_{n=0}^{\infty} x_2(t-n) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

14) Linear. Assuming that only a finite number of jumps occur in the interval  $(-\infty, t]$  and that the magnitude of these jumps is finite so that the algebraic sum is well defined, we obtain

$$y(t) = T[\alpha x(t)] = \sum_{n=1}^N \alpha J_x(t_n) = \alpha \sum_{n=1}^N J_x(t_n) = \alpha T[x(t)]$$

where  $N$  is the number of jumps in  $(-\infty, t]$  and  $J_x(t_n)$  is the value of the jump at time instant  $t_n$ , that is

$$J_x(t_n) = \lim_{\epsilon \rightarrow 0} (x(t_n + \epsilon) - x(t_n - \epsilon))$$

For  $x(t) = x_1(t) + x_2(t)$  we can assume that  $x_1(t)$ ,  $x_2(t)$  and  $x(t)$  have the same number of jumps and at the same positions. This is true since we can always add new jumps of magnitude zero to the already existing ones. Then for each  $t_n$ ,  $J_x(t_n) = J_{x_1}(t_n) + J_{x_2}(t_n)$  and

$$y(t) = \sum_{n=1}^N J_x(t_n) = \sum_{n=1}^N J_{x_1}(t_n) + \sum_{n=1}^N J_{x_2}(t_n)$$

so that the system is additive.

### Problem 2.17

Only if ( $\implies$ )

for some  $x_1(t), x_2(t)$ . To see this let  $x_2(t) = c$  (a constant signal). Then

$$T[x_1(t) + x_2(t)] = \frac{(x_1(t) + c)^2}{x_1'(t)} = \frac{x_1^2(t) + 2cx_1(t) + c^2}{x_1'(t)}$$

and

$$T[x_1(t)] + T[x_2(t)] = \frac{x_1^2(t)}{x_1'(t)}$$

Thus  $T[x_1(t) + x_2(t)] \neq T[x_1(t)] + T[x_2(t)]$  unless  $c = 0$ . Hence the system is nonlinear since the additive property has to hold for every  $x_1(t)$  and  $x_2(t)$ .

As another example of a system that is homogeneous but non linear is the system described by

$$T[x(t)] = \begin{cases} x(t) + x(t-1) & x(t)x(t-1) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Clearly  $T[\alpha x(t)] = \alpha T[x(t)]$  but  $T[x_1(t) + x_2(t)] \neq T[x_1(t)] + T[x_2(t)]$

### Problem 2.21

Any zero input signal can be written as  $0 \cdot x(t)$  with  $x(t)$  an arbitrary signal. Then, the response of the linear system is  $y(t) = \mathcal{L}[0 \cdot x(t)]$  and since the system is homogeneous (linear system) we obtain

$$y(t) = \mathcal{L}[0 \cdot x(t)] = 0 \cdot \mathcal{L}[x(t)] = 0$$

Thus the response of the linear system is identically zero.

### Problem 2.22

For the system to be linear we must have

$$T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)]$$

for every  $\alpha, \beta$  and  $x(t)$ 's.

$$\begin{aligned} T[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t)) \cos(2\pi f_0 t) \\ &= \alpha x_1(t) \cos(2\pi f_0 t) + \beta x_2(t) \cos(2\pi f_0 t) \\ &= \alpha T[x_1(t)] + \beta T[x_2(t)] \end{aligned}$$

Thus the system is linear. In order for the system to be time-invariant the response to  $x(t - t_0)$  should be  $y(t - t_0)$  where  $y(t)$  is the response of the system to  $x(t)$ . Clearly  $y(t - t_0) = x(t - t_0) \cos(2\pi f_0(t - t_0))$  and the response of the system to  $x(t - t_0)$  is  $y'(t) = x(t - t_0) \cos(2\pi f_0 t)$ . Since  $\cos(2\pi f_0(t - t_0))$  is not equal to  $\cos(2\pi f_0 t)$  for all  $t, t_0$  we conclude that  $y'(t) \neq y(t - t_0)$  and thus the system is time-variant.

### Problem 2.23

1) False. For if  $T_1[x(t)] = x^3(t)$  and  $T_2[x(t)] = x^{1/3}(t)$  then the cascade of the two systems is the identity system  $T[x(t)] = x(t)$  which is known to be linear. However, both  $T_1[\cdot]$  and  $T_2[\cdot]$  are nonlinear.

2) False. For if

$$T_1[x(t)] = \begin{cases} tx(t) & t \neq 0 \\ 0 & t = 0 \end{cases} \quad T_2[x(t)] = \begin{cases} \frac{1}{t}x(t) & t \neq 0 \\ 0 & t = 0 \end{cases}$$

Then  $T_2[T_1[x(t)]] = x(t)$  and the system which is the cascade of  $T_1[\cdot]$  followed by  $T_2[\cdot]$  is time-invariant, whereas both  $T_1[\cdot]$  and  $T_2[\cdot]$  are time variant.

3) False. Consider the system

$$y(t) = T[x(t)] = \begin{cases} x(t) & t \geq 0 \\ 1 & t < 0 \end{cases}$$

Then the output of the system  $y(t)$  depends only on the input  $x(\tau)$  for  $\tau \leq t$ . This means that the system is causal. However the response to a causal signal,  $x(t) = 0$  for  $t \leq 0$ , is nonzero for negative values of  $t$  and thus it is not causal.

### Problem 2.24

1) Time invariant: The response to  $x(t - t_0)$  is  $2x(t - t_0) + 3$  which is  $y(t - t_0)$ .

2) Time varying: The response to  $x(t - t_0)$  is  $(t + 2)x(t - t_0)$  but  $y(t - t_0) = (t - t_0 + 2)x(t - t_0)$ , obviously the two are not equal.

3) Time varying: The response to  $x(t - t_0)$  is  $t + x(t - t_0)$  whereas  $y(t - t_0) = x(t - t_0) + t - t_0$ .

4) Time-varying system. The response  $y(t - t_0)$  is equal to  $x(-(t - t_0)) = x(-t + t_0)$ . However the response of the system to  $x(t - t_0)$  is  $z(t) = x(-t - t_0)$  which is not equal to  $y(t - t_0)$ .

5) Time-varying system. Clearly

$$y(t) = x(t)u_{-1}(t) \implies y(t - t_0) = x(t - t_0)u_{-1}(t - t_0)$$

However, the response of the system to  $x(t - t_0)$  is  $z(t) = x(t - t_0)u_{-1}(t)$  which is not equal to  $y(t - t_0)$ .

6) Time-varying system. Clearly

$$y(t) = x(t)\delta(t) \implies y(t - t_0) = x(t - t_0)\delta(t - t_0) = x(0)\delta(t - t_0)$$

However, the response of the system to  $x(t - t_0)$  is  $z(t) = x(t - t_0)\delta(t) = x(-t_0)\delta(t)$  which is different with  $y(t - t_0)$ .

7) Time-varying system. It is true that  $y(t - t_0)$  is the response to  $x(t - t_0)$  for  $t_0 = mT$ . However, for the system to be time-invariant this relation should hold for every  $t_0$  which is not the case.

8) Time-invariant system. Clearly

$$y(t) = \int_{-\infty}^t x(\tau)d\tau \implies y(t - t_0) = \int_{-\infty}^{t-t_0} x(\tau)d\tau$$

The response of the system to  $x(t - t_0)$  is

$$z(t) = \int_{-\infty}^t x(\tau - t_0)d\tau = \int_{-\infty}^{t-t_0} x(v)dv = y(t - t_0)$$

where we have used the change of variable  $v = \tau - t_0$ .

9) Time-invariant system. Writing  $y(t)$  as  $\sum_{n=-\infty}^{\infty} x(t-n)$  we get

$$y(t-t_0) = \sum_{n=-\infty}^{\infty} x(t-t_0-n) = T[x(t-t_0)]$$

10) Time-invariant system. The response of the system is simply  $y(t) = \text{sgn}(x(t))$ . Thus  $y(t-t_0) = \text{sgn}(x(t-t_0))$  is the response of the system to  $x(t-t_0)$ .

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### Problem 2.25

The differentiator is a LTI system (see examples 2.19 and 2.1.21 in book). It is true that the output of a system which is the cascade of two LTI systems does not depend on the order of the systems. This can be easily seen by the commutative property of the convolution

$$h_1(t) \star h_2(t) = h_2(t) \star h_1(t)$$

Let  $h_1(t)$  be the impulse response of a differentiator, and let  $y(t)$  be the output of the system  $h_2(t)$  with input  $x(t)$ . Then,

$$\begin{aligned} z(t) &= h_2(t) \star x'(t) = h_2(t) \star (h_1(t) \star x(t)) \\ &= h_2(t) \star h_1(t) \star x(t) = h_1(t) \star h_2(t) \star x(t) \\ &= h_1(t) \star y(t) = y'(t) \end{aligned}$$

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### Problem 2.26

The integrator is a LTI system (why?). It is true that the output of a system which is the cascade of two LTI systems does not depend on the order of the systems. This can be easily seen by the commutative property of the convolution

$$h_1(t) \star h_2(t) = h_2(t) \star h_1(t)$$

Let  $h_1(t)$  be the impulse response of an integrator, and let  $y(t)$  be the output of the system  $h_2(t)$  with input  $x(t)$ . Then,

$$\begin{aligned} z(t) &= h_2(t) \star \int_{-\infty}^t x(\tau) d\tau = h_2(t) \star (h_1(t) \star x(t)) \\ &= h_2(t) \star h_1(t) \star x(t) = h_1(t) \star h_2(t) \star x(t) \\ &= h_1(t) \star y(t) = \int_{-\infty}^t y(\tau) d\tau \end{aligned}$$

---

### Problem 2.27

The output of a LTI system is the convolution of the input with the impulse response of the system. Thus,

$$\delta(t) = \int_{-\infty}^{\infty} h(\tau) e^{-\alpha(t-\tau)} u_{-1}(t-\tau) d\tau = \int_{-\infty}^t h(\tau) e^{-\alpha(t-\tau)} d\tau$$

Differentiating both sides with respect to  $t$  we obtain

$$\begin{aligned}\delta'(t) &= (-\alpha)e^{-\alpha t} \int_{-\infty}^t h(\tau)e^{\alpha\tau} d\tau + e^{-\alpha t} \frac{d}{dt} \left[ \int_{-\infty}^t h(\tau)e^{\alpha\tau} d\tau \right] \\ &= (-\alpha)\delta(t) + e^{-\alpha t} h(t)e^{\alpha t} = (-\alpha)\delta(t) + h(t)\end{aligned}$$

Thus

$$h(t) = \alpha\delta(t) + \delta'(t)$$

The response of the system to the input  $x(t)$  is

$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty} x(\tau) [\alpha\delta(t-\tau) + \delta'(t-\tau)] d\tau \\ &= \alpha \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)\delta'(t-\tau)d\tau \\ &= \alpha x(t) + \frac{d}{dt}x(t)\end{aligned}$$

### Problem 2.28

For the system to be causal the output at the time instant  $t_0$  should depend only on  $x(t)$  for  $t \leq t_0$ .

$$y(t_0) = \frac{1}{2T} \int_{t_0-T}^{t_0+T} x(\tau)d\tau = \frac{1}{2T} \int_{t_0-T}^{t_0} x(\tau)d\tau + \frac{1}{2T} \int_{t_0}^{t_0+T} x(\tau)d\tau$$

We observe that the second integral on the right side of the equation depends on values of  $x(\tau)$  for  $\tau$  greater than  $t_0$ . Thus the system is non causal.

### Problem 2.29

Consider the system

$$y(t) = T[x(t)] = \begin{cases} x(t) & x(t) \neq 0 \\ 1 & x(t) = 0 \end{cases}$$

This system is causal since the output at the time instant  $t$  depends only on values of  $x(\tau)$  for  $\tau \leq t$  (actually it depends only on the value of  $x(\tau)$  for  $\tau = t$ , a stronger condition.) However, the response of the system to the impulse signal  $\delta(t)$  is one for  $t < 0$  so that the impulse response of the system is nonzero for  $t < 0$ .

### Problem 2.30

1. Noncausal: Since for  $t < 0$  we do not have  $\text{sinc}(t) = 0$ .
2. This is a rectangular signal of width 6 centered at  $t_0 = 3$ , for negative  $t$ 's it is zero, therefore the system is causal.
3. The system is causal since for negative  $t$ 's  $h(t) = 0$ .

---

**Problem 2.33**

The response of the system to the signal  $\alpha x_1(t) + \beta x_2(t)$  is

$$y_1(t) = \int_{t-T}^t (\alpha x_1(\tau) + \beta x_2(\tau)) d\tau = \alpha \int_{t-T}^t x_1(\tau) d\tau + \beta \int_{t-T}^t x_2(\tau) d\tau$$

Thus the system is linear. The response to  $x(t - t_0)$  is

$$y_1(t) = \int_{t-T}^t x(\tau - t_0) d\tau = \int_{t-t_0-T}^{t-t_0} x(v) dv = y(t - t_0)$$

where we have used the change of variables  $v = \tau - t_0$ . Thus the system is time invariant. The impulse response is obtained by applying an impulse at the input.

$$h(t) = \int_{t-T}^t \delta(\tau) d\tau = \int_{-\infty}^t \delta(\tau) d\tau - \int_{-\infty}^{t-T} \delta(\tau) d\tau = u_{-1}(t) - u_{-1}(t - T)$$

---

**Problem 2.34**

1)

$$\begin{aligned} e^{-t} u_{-1}(t) \star e^{-t} u_{-1}(t) &= \int_{-\infty}^{\infty} e^{-\tau} u_{-1}(\tau) e^{-(t-\tau)} u_{-1}(t-\tau) d\tau = \int_0^t e^{-\tau} d\tau \\ &= \begin{cases} t e^{-t} & t > 0 \\ 0 & t < 0 \end{cases} \end{aligned}$$

2)

$$\begin{aligned} e^{-t} u_{-1}(t) \star u_{-1}(t) &= \int_{-\infty}^{\infty} u_{-1}(\tau) e^{-(t-\tau)} u_{-1}(t-\tau) d\tau \\ &= \int_0^t e^{-(t-\tau)} d\tau = e^{-t} e^{\tau} \Big|_0^t \\ &= \begin{cases} 1 - e^{-t} & t > 0 \\ 0 & t < 0 \end{cases} \end{aligned}$$

3)

$$x(t) = \Pi(t) \star \Lambda(t) = \int_{-\infty}^{\infty} \Pi(\theta) \Lambda(t - \theta) d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(t - \theta) d\theta = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \Lambda(v) dv$$

$$\begin{aligned}
t \leq -\frac{3}{2} &\implies x(t) = 0 \\
-\frac{3}{2} < t \leq -\frac{1}{2} &\implies x(t) = \int_{-1}^{t+\frac{1}{2}} (v+1)dv = \left(\frac{1}{2}v^2 + v\right)\Big|_{-1}^{t+\frac{1}{2}} = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{8} \\
-\frac{1}{2} < t \leq \frac{1}{2} &\implies x(t) = \int_{t-\frac{1}{2}}^0 (v+1)dv + \int_0^{t+\frac{1}{2}} (-v+1)dv \\
&= \left(\frac{1}{2}v^2 + v\right)\Big|_{t-\frac{1}{2}}^0 + \left(-\frac{1}{2}v^2 + v\right)\Big|_0^{t+\frac{1}{2}} = -t^2 + \frac{3}{4} \\
\frac{1}{2} < t \leq \frac{3}{2} &\implies x(t) = \int_{t-\frac{1}{2}}^1 (-v+1)dv = \left(-\frac{1}{2}v^2 + v\right)\Big|_{t-\frac{1}{2}}^1 = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{9}{8} \\
\frac{3}{2} < t &\implies x(t) = 0
\end{aligned}$$

Thus,

$$x(t) = \begin{cases} 0 & t \leq -\frac{3}{2} \\ \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{8} & -\frac{3}{2} < t \leq -\frac{1}{2} \\ -t^2 + \frac{3}{4} & -\frac{1}{2} < t \leq \frac{1}{2} \\ \frac{1}{2}t^2 - \frac{3}{2}t + \frac{9}{8} & \frac{1}{2} < t \leq \frac{3}{2} \\ 0 & \frac{3}{2} < t \end{cases}$$

4)

$$x(t) = \Lambda(t) \operatorname{sgn}(t) \star u_{-1}(t) = \int_{-\infty}^t \Lambda(\tau) \operatorname{sgn}(\tau) d\tau$$

$$\begin{aligned}
t \leq -1 &\implies x(t) = 0 \\
-1 < t \leq 0 &\implies x(t) = \int_{-\infty}^t (-v-1)dv = \left(-\frac{1}{2}v^2 - v\right)\Big|_{-1}^t = -\frac{1}{2}t^2 - t - \frac{1}{2} \\
0 < t \leq 1 &\implies x(t) = -\frac{1}{2} + \int_0^t (-v+1)dv = -\frac{1}{2} + \left(-\frac{1}{2}v^2 + v\right)\Big|_0^t = -\frac{1}{2}t^2 + t - \frac{1}{2} \\
1 < t &\implies x(t) = 0
\end{aligned}$$

5)

$$x(t) = \Lambda(t) \star \operatorname{sgn}(t) = \int_{-\infty}^{\infty} \Lambda(\tau) \operatorname{sgn}(t-\tau) d\tau = \int_{-\infty}^t \Lambda(\tau) d\tau - \int_t^{\infty} \Lambda(\tau) d\tau = x_1(t) - x_2(t)$$

$$\begin{aligned}
t \leq -1 &\implies x_1(t) = 0, \quad x_2(t) = 1 \text{ and } x(t) = -1 \\
-1 \leq t \leq 0 &\implies x_1(t) = \int_{-1}^t (v+1)dv = \left(\frac{1}{2}v^2 + v\right)\Big|_{-1}^t = \frac{1}{2}t^2 + t + \frac{1}{2} \\
&x_2(t) = \int_t^0 (v+1)dv + \int_0^1 (-v+1)dv = \left(\frac{1}{2}v^2 + v\right)\Big|_t^0 + \frac{1}{2} \\
&= -\frac{1}{2}t^2 - t + \frac{1}{2} \\
&x(t) = x_1(t) - x_2(t) = t^2 + 2t \\
0 \leq t \leq 1 &\implies x_1(t) = \frac{1}{2} \int_0^t (-v+1)dv = \frac{1}{2} + \left(-\frac{1}{2}v^2 + v\right)\Big|_0^t = -\frac{1}{2}t^2 + t + \frac{1}{2} \\
&x_2(t) = \int_t^1 (-v+1)dv = \left(-\frac{1}{2}v^2 + v\right)\Big|_t^1 = \frac{1}{2}t^2 - t + \frac{1}{2} \\
&x(t) = x_1(t) - x_2(t) = -t^2 + 2t \\
1 \leq t &\implies x_1(t) = 1, \quad x_2(t) = 0 \text{ and } x(t) = 1
\end{aligned}$$

6)

$$x(t) = \Lambda(t)u_{-1}(t) \star \Pi(t) = \int_0^{\infty} \Lambda(\tau)\Pi(t-\tau)d\tau$$

Note that  $\Pi(t-\tau) = 1$  for  $|t-\tau| < 1/2$  and zero otherwise. Thus

$$\begin{aligned}
t < -\frac{1}{2} &\implies x(t) = 0 \\
-\frac{1}{2} \leq t \leq \frac{1}{2} &\implies x(t) = \int_0^{t+\frac{1}{2}} \Lambda(\tau)d\tau = \left(-\frac{1}{2}v^2 + v\right)\Big|_0^{t+\frac{1}{2}} = -\frac{1}{2}t^2 + \frac{1}{2}t + \frac{3}{8} \\
\frac{1}{2} \leq t \leq \frac{3}{2} &\implies x(t) = \int_{t-\frac{1}{2}}^1 \Lambda(\tau)d\tau = \left(-\frac{1}{2}v^2 + v\right)\Big|_{t-\frac{1}{2}}^1 = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{9}{8} \\
\frac{3}{2} < t &\implies x(t) = 0
\end{aligned}$$

### Problem 2.35

The output of a LTI system with impulse response  $h(t)$  is

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Using the first formula for the convolution and observing that  $h(\tau) = 0, \tau < 0$  we obtain

$$y(t) = \int_{-\infty}^0 x(t-\tau)h(\tau)d\tau + \int_0^{\infty} x(t-\tau)h(\tau)d\tau = \int_0^{\infty} x(t-\tau)h(\tau)d\tau$$

Using the second formula for the convolution and writing

$$y(t) = \int_{-\infty}^t x(\tau)h(t-\tau)d\tau + \int_t^{\infty} x(\tau)h(t-\tau)d\tau$$

Equality holds if  $x(t) = -\alpha y(t)$  a.e. for some complex  $\alpha$ .

**Problem 2.38**

1)

$$\begin{aligned}
 \epsilon^2 &= \int_{-\infty}^{\infty} \left| x(t) - \sum_{i=1}^N \alpha_i \phi_i(t) \right|^2 dt \\
 &= \int_{-\infty}^{\infty} \left( x(t) - \sum_{i=1}^N \alpha_i \phi_i(t) \right) \left( x^*(t) - \sum_{j=1}^N \alpha_j^* \phi_j^*(t) \right) dt \\
 &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^N \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^N \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt \\
 &\quad + \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j^* \int_{-\infty}^{\infty} \phi_i(t) \phi_j^*(t) dt \\
 &= \int_{-\infty}^{\infty} |x(t)|^2 dt + \sum_{i=1}^N |\alpha_i|^2 - \sum_{i=1}^N \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^N \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt
 \end{aligned}$$

Completing the square in terms of  $\alpha_i$  we obtain

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^N \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 + \sum_{i=1}^N \left| \alpha_i - \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2$$

The first two terms are independent of  $\alpha$ 's and the last term is always positive. Therefore the minimum is achieved for

$$\alpha_i = \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt$$

which causes the last term to vanish.

2) With this choice of  $\alpha_i$ 's

$$\begin{aligned}
 \epsilon^2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^N \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 \\
 &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^N |\alpha_i|^2
 \end{aligned}$$

**Problem 2.39**

1) Using Euler's relation we have

$$\begin{aligned}
 x_1(t) &= \cos(2\pi t) + \cos(4\pi t) \\
 &= \frac{1}{2} (e^{j2\pi t} + e^{-j2\pi t} + e^{j4\pi t} + e^{-j4\pi t})
 \end{aligned}$$

Therefore for  $n = \pm 1, \pm 2$ ,  $x_{1,n} = \frac{1}{2}$  and for all other values of  $n$ ,  $x_{1,n} = 0$ .

2) Using Euler's relation we have

$$\begin{aligned} x_2(t) &= \cos(2\pi t) - \cos(4\pi t + \pi/3) \\ &= \frac{1}{2} (e^{j2\pi t} + e^{-j2\pi t} - e^{j(4\pi t + \pi/3)} - e^{-j(4\pi t + \pi/3)}) \\ &= \frac{1}{2} e^{j2\pi t} + \frac{1}{2} e^{-j2\pi t} + \frac{1}{2} e^{-j2\pi/3} e^{j4\pi t} + \frac{1}{2} e^{j2\pi/3} e^{-j4\pi t} \end{aligned}$$

from this we conclude that  $x_{2,\pm 1} = \frac{1}{2}$  and  $x_{2,2} = x_{2,-2}^* = \frac{1}{2} e^{-j2\pi/3}$ , and for all other values of  $n$ ,  $x_{2,n} = 0$ .

3) We have  $x_3(t) = 2 \cos(2\pi t) - \sin(4\pi t) = 2 \cos(2\pi t) + \cos(4\pi t + \pi/2)$ . Using Euler's relation as in parts 1 and 2 we see that  $x_{3,\pm 1} = 1$  and  $x_{3,2} = x_{3,-2}^* = j$ , and for all other values of  $n$ ,  $x_{3,n} = 0$ .

4) The signal  $x_4(t)$  is periodic with period  $T_0 = 2$ . Thus

$$\begin{aligned} x_{4,n} &= \frac{1}{2} \int_{-1}^1 \Lambda(t) e^{-j2\pi \frac{n}{2} t} dt = \frac{1}{2} \int_{-1}^1 \Lambda(t) e^{-j\pi n t} dt \\ &= \frac{1}{2} \int_{-1}^0 (t+1) e^{-j\pi n t} dt + \frac{1}{2} \int_0^1 (-t+1) e^{-j\pi n t} dt \\ &= \frac{1}{2} \left( \frac{j}{\pi n} t e^{-j\pi n t} + \frac{1}{\pi^2 n^2} e^{-j\pi n t} \right) \Big|_{-1}^0 + \frac{j}{2\pi n} e^{-j\pi n t} \Big|_{-1}^0 \\ &\quad - \frac{1}{2} \left( \frac{j}{\pi n} t e^{-j\pi n t} + \frac{1}{\pi^2 n^2} e^{-j\pi n t} \right) \Big|_0^1 + \frac{j}{2\pi n} e^{-j\pi n t} \Big|_0^1 \\ &= \frac{1}{\pi^2 n^2} - \frac{1}{2\pi^2 n^2} (e^{j\pi n} + e^{-j\pi n}) = \frac{1}{\pi^2 n^2} (1 - \cos(\pi n)) \end{aligned}$$

When  $n = 0$  then

$$x_{4,0} = \frac{1}{2} \int_{-1}^1 \Lambda(t) dt = \frac{1}{2}$$

Thus

$$x_4(t) = \frac{1}{2} + 2 \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} (1 - \cos(\pi n)) \cos(\pi n t)$$

5)  $x_5(t) = 1$ . It follows then that  $x_{5,0} = 1$  and  $x_{5,n} = 0$ ,  $\forall n \neq 0$ .

6) The signal is periodic with period  $T_0 = 1$ . Thus

$$\begin{aligned} x_{6,n} &= \frac{1}{T_0} \int_0^{T_0} e^t e^{-j2\pi n t} dt = \int_0^1 e^{(-j2\pi n + 1)t} dt \\ &= \frac{1}{-j2\pi n + 1} e^{(-j2\pi n + 1)t} \Big|_0^1 = \frac{e^{(-j2\pi n + 1)} - 1}{-j2\pi n + 1} \\ &= \frac{e - 1}{1 - j2\pi n} = \frac{e - 1}{\sqrt{1 + 4\pi^2 n^2}} (1 + j2\pi n) \end{aligned}$$

7) The signal  $\cos(t)$  is periodic with period  $T_1 = 2\pi$  whereas  $\cos(2.5t)$  is periodic with period  $T_2 = 0.8\pi$ . It follows then that  $\cos(t) + \cos(2.5t)$  is periodic with period  $T = 4\pi$ . The trigonometric Fourier series of the even signal  $\cos(t) + \cos(2.5t)$  is

$$\begin{aligned}\cos(t) + \cos(2.5t) &= \sum_{n=1}^{\infty} \alpha_n \cos(2\pi \frac{n}{T_0} t) \\ &= \sum_{n=1}^{\infty} \alpha_n \cos(\frac{n}{2} t)\end{aligned}$$

By equating the coefficients of  $\cos(\frac{n}{2}t)$  of both sides we observe that  $a_n = 0$  for all  $n$  unless  $n = 2, 5$  in which case  $a_2 = a_5 = 1$ . Hence  $x_{7,2} = x_{7,5} = \frac{1}{2}$  and  $x_{7,n} = 0$  for all other values of  $n$ .

8) The signal  $x_8(t)$  is periodic with period  $T_0 = 1$ . For  $n = 0$

$$x_{8,0} = \int_0^1 (-t + 1) dt = \left(-\frac{1}{2}t^2 + t\right)\Big|_0^1 = \frac{1}{2}$$

For  $n \neq 0$

$$\begin{aligned}x_{8,n} &= \int_0^1 (-t + 1)e^{-j2\pi nt} dt \\ &= -\left(\frac{j}{2\pi n}te^{-j2\pi nt} + \frac{1}{4\pi^2 n^2}e^{-j2\pi nt}\right)\Big|_0^1 + \frac{j}{2\pi n}e^{-j2\pi nt}\Big|_0^1 \\ &= -\frac{j}{2\pi n}\end{aligned}$$

Thus,

$$x_8(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin 2\pi nt$$

9) The signal  $x_9(t)$  is periodic with period  $T_0 = 2T$ . We can write  $x_9(t)$  as

$$\begin{aligned}x_9(t) &= \sum_{n=-\infty}^{\infty} \delta(t - n2T) - \sum_{n=-\infty}^{\infty} \delta(t - T - n2T) \\ &= \frac{1}{2T} \sum_{n=-\infty}^{\infty} e^{j\pi \frac{n}{T}t} - \frac{1}{2T} \sum_{n=-\infty}^{\infty} e^{j\pi \frac{n}{T}(t-T)} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2T} (1 - e^{-j\pi n}) e^{j2\pi \frac{n}{2T}t}\end{aligned}$$

However, this is the Fourier series expansion of  $x_9(t)$  and we identify  $x_{9,n}$  as

$$x_{9,n} = \frac{1}{2T} (1 - e^{-j\pi n}) = \frac{1}{2T} (1 - (-1)^n) = \begin{cases} 0 & n \text{ even} \\ \frac{1}{T} & n \text{ odd} \end{cases}$$

10) The signal is periodic with period  $T$ . Thus,

$$\begin{aligned} x_{7,n} &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta'(t) e^{-j2\pi \frac{n}{T} t} dt \\ &= \frac{1}{T} (-1) \frac{d}{dt} e^{-j2\pi \frac{n}{T} t} \Big|_{t=0} = \frac{j2\pi n}{T^2} \end{aligned}$$

11) The signal  $x_8(t)$  is real even and periodic with period  $T_0 = \frac{1}{2f_0}$ . Hence,  $x_{8,n} = a_{8,n}/2$  or

$$\begin{aligned} x_{8,n} &= 2f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t) \cos(2\pi n 2f_0 t) dt \\ &= f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+2n)t) dt + f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-2n)t) dt \\ &= \frac{1}{2\pi(1+2n)} \sin(2\pi f_0(1+2n)t) \Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-2n)} \sin(2\pi f_0(1-2n)t) \Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\ &= \frac{(-1)^n}{\pi} \left[ \frac{1}{(1+2n)} + \frac{1}{(1-2n)} \right] \end{aligned}$$

12) The signal  $x_9(t) = \cos(2\pi f_0 t) + |\cos(2\pi f_0 t)|$  is even and periodic with period  $T_0 = 1/f_0$ . It is equal to  $2\cos(2\pi f_0 t)$  in the interval  $[-\frac{1}{4f_0}, \frac{1}{4f_0}]$  and zero in the interval  $[\frac{1}{4f_0}, \frac{3}{4f_0}]$ . Thus

$$\begin{aligned} x_{9,n} &= 2f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t) \cos(2\pi n f_0 t) dt \\ &= f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+n)t) dt + f_0 \int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-n)t) dt \\ &= \frac{1}{2\pi(1+n)} \sin(2\pi f_0(1+n)t) \Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-n)} \sin(2\pi f_0(1-n)t) \Big|_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\ &= \frac{1}{\pi(1+n)} \sin\left(\frac{\pi}{2}(1+n)\right) + \frac{1}{\pi(1-n)} \sin\left(\frac{\pi}{2}(1-n)\right) \end{aligned}$$

Thus  $x_{9,n}$  is zero for odd values of  $n$  unless  $n = \pm 1$  in which case  $x_{9,\pm 1} = \frac{1}{2}$ . When  $n$  is even ( $n = 2l$ ) then

$$x_{9,2l} = \frac{(-1)^l}{\pi} \left[ \frac{1}{1+2l} + \frac{1}{1-2l} \right]$$

#### Problem 2.40

It follows directly from the uniqueness of the decomposition of a real signal in an even and odd part. Nevertheless for a real periodic signal

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(2\pi \frac{n}{T_0} t) + b_n \sin(2\pi \frac{n}{T_0} t) \right]$$