Lecture:July 4th, 2006 Ayman AL Khatib, Ellia Lim

Final Exam Review

Review Topics:

1. **Random experiments, Sample Space, Events**

2. **Probability of axioms**

   i $\forall A \subseteq S, P[A] \geq 0$

   ii $P[S] = 1$

   iii For any disjoint events $A_1, A_2, ...A_n$

$$P\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} P[A_i]$$

3. **Random Variables vs. Events**
   R.V. X, where $X = 1$ or $X > 1$ is an event about X.
   $P_x(x)$ is a function of x, where x can be treated as an event.

4. **Conditional Probability**
   Let A and B be some arbitrary events, the probability of A given B can be expressed as:

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

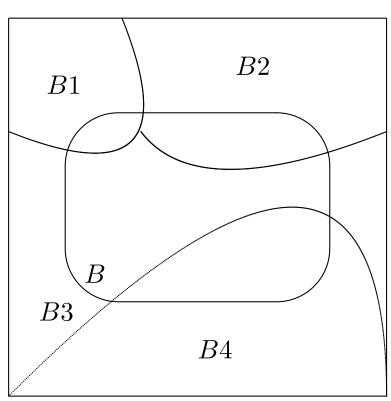
   If X and Y are random variables the conditional probability in discrete form is given as:

$$P_{X|Y}(x|y) = \frac{P_{XY}(x, y)}{P_Y(y)}$$

   If X and Y are random variables the conditional probability in continuous form is given as:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

5. **Total Probability and Bayes' Rule**



Given $A \subseteq S, B_{i\infty}$ is a partition of S, the total probability theorem states:

$$P[A] = \sum_{i=1}^{\infty} P[A|B_i]P[B_i]$$

Bayes' theorem states:

$$P[B_3|A] = \frac{P[A|B_3]P[B_3]}{\sum_{i=1}^{\infty} P[A|B_i]P[B_i]}$$

6. **Total Probability and Bayes' Rule in R.V.**
In discrete form the the total probability theorem states:

$$\sum_x P_{XY}(x,y) = P_Y(y)$$

In continuous form:

$$f_Y(y) = \int_{+\infty}^{-\infty} f_{XY}(x,y)dx = \int_{+\infty}^{-\infty} f_{X|Y}(y|x)f_x(x)dx$$

In discrete form the the Bayes' Rule states:

$$P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x)P_x(x)}{\sum_x P_{Y|X}(x)P_x(x)}$$

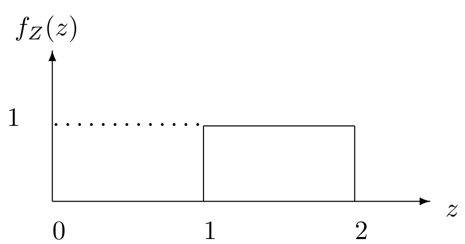
In continuous form:

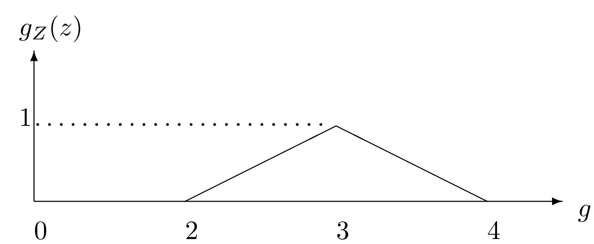
$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_x(x)}{\int_{-\infty}^{+\infty} f_{Y|X}(x)f_x(x)}$$

Example 1 `Please refer to Question 7 in 2005 final exam.`
$P[cancer] = 0.1$
$P[+|cancer] = 0.9$
$P[+|nocancer] = 0.3$
Find the probability that Jennifer has cancer given her test is "+"?
Solution:

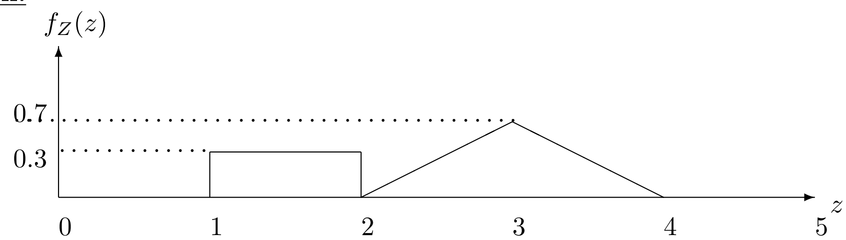
$$\begin{aligned} P[cancer|"+"] &= \frac{P[cancer AND "+"]}{P["+"]} \\ &= \frac{P["+"|cancer]P[cancer]}{P["+"|cancer]P[cancer] + P["+"|nocancer]P[nocancer]} \\ &= \frac{(0.9)(.1)}{(0.9)(0.1) + (0.3)(0.9)} \\ &= 0.25 \end{aligned}$$

Example 2 Toss a coin with P[H]=0.3, if H draw random z from f(z),else if T draw z from g(z).





Solution:



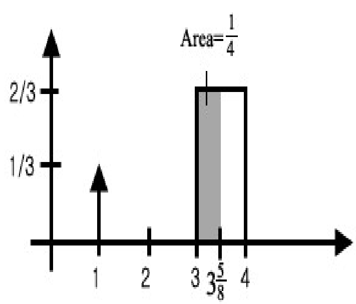
$f_Z(z) = f_{Z|H}P[H] + f_{Z|T}P[T]$

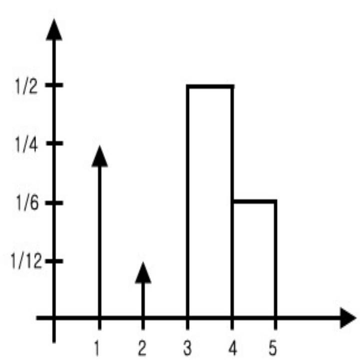
Example 3 `2005 Final Q3`

Suppose that you are only allowed to perform the following two random experiments.
$Exp1 : Generatesarandomnumberfollowingdistribution given Figure$
$Exp2 : Tossingafaircoin. You are allowed to perform the two random experiments as many times as you want.$

1. By performing the two random experiments and any arithmetic computation, generate a random variable with distribution given in figure. Describe you procedure(flow-chart or pseudo-code is recommended).

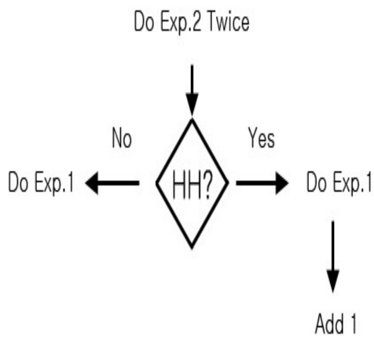
2. Suppose that now you are not allowed to do Experiment 2 but still allowed to do Experiment 1 and any arithmetic computation. Can you still generate a random variable with distribution in Figure 2? Sketch your procedure if you can, or explain why this is impossible.



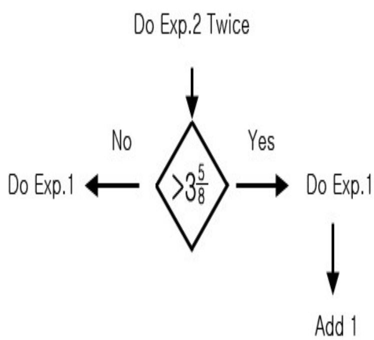


Solution:

1.



2.



Example 4 `2006 Midterm Q2`

Due to some problem in the manufacturing process of an electrical thermometer, there is 10% probability that a thermometer has defect. Let T denote the true temperature. It is known that if a thermometer has no defect, it outputs a temperature reading in set T - 0.1, T, T + 0.1, where the probability of outputting T -0.1, T, and T +0.1 are respectively 0.1, 0.8, and 0.1. It is also known that when a thermometer has defect, it outputs a temperature reading in set T, T +0.1, T +1, where the probabilities of outputting T, T + 0.1, and T + 1 are respectively 0.1, 0.3 and 0.6.

Suppose that the true temperature T is 20, and I will pick a thermometer at random to measure temperature. Let X be the reading of the thermometer.

If you are told that the temperature reading X is not lower than 20.1, find the PDF and CDF of X given this information.

Solution:

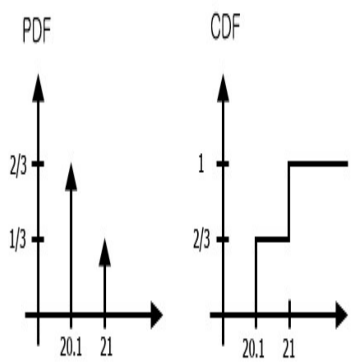
$P[defect] = 0.1$
$T = 20$

$nodefect : P[T-1] = 0.1, P[T] = 0.8, P[T+1] = 0.1$
$defect : P[T] = 0.1, P[T+0.1] = 0.3, P[T+1] = 0.6$

Without observing X:
P[X=19.9 and no defect]= 0.9*0.1 = 0.09
P[X=20 and no defect]= 0.9*0.8 = 0.72
P[X=20.1 and defect]= 0.9*0.1 = 0.09
P[X=20 and defect]= 0.1*0.1 = 0.01
P[X=20.1 and defect]= 0.1*0.3 = 0.0
P[X=21 and defect]= 0.1*0.6 = 0.06

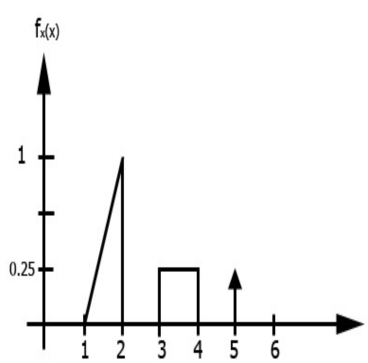
$P[X = 20.1] = 0.03 + 0.09 = 0.12$
$P[X = 21] = 0.06$

Conditioned on $X \geq 20.1$: $P_{X|X\geq 20.1}(x)= \begin{cases} \frac{1}{3}, & \text{X=21,} \\ \frac{2}{3}, & \text{X=20.1} \end{cases}$



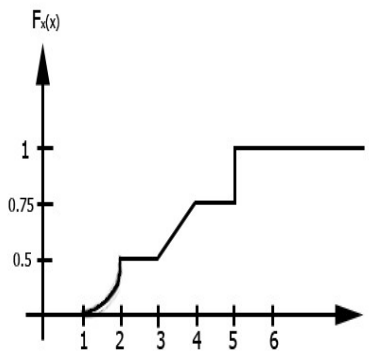
Example 5

Given PDF, find CDF.



If x$\leq 1, F_X(x) = 0$
$1 \leq x < 2$,F$_X(x) = \int_{-\infty}^{t} f_x(t)\, dt. = \frac{x-1}{2}$
$F_X(2) = \frac{1}{2}$
$F_X(4) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$F_X(5) = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$



7. **Independence**

Independence of Events:
1. Events A and B are said to be independent if P[A$\cap B] = P[A]P[B]$
2.$Events A_1, A_2, ..., A_n are said to be independent if \forall k \in 2, 3, ..., n, and any A_i 1, A_i 2, ..., A_{ik}$
P$[[\bigcap_{j=1}^{k} A_{ij}] = \prod_{j=1}^{k} P[A_{ij}]]$

Independence of Random Variables:
1. RV's X and Y are said to be independent if
P$_{XY}(x, y) = P_X(x)P_Y(y) or f_{XY}(x, y) = f_X(x)f_Y(y)$
2.$RV's X_1, X_2, ..., X_m are said to be independent if$
$P_{X1}, P_{X2}, ..., P_{Xm}(x_1, x_2, ..., x_m) = [\prod_{i=1}^{m} P_{Xi}(xi)] or$
$f_{X1}, f_[X2, ..., f_[Xm(x_1, x_2, ..., x_m) = [\prod_{i=1}^{m} f_{Xi}(xi)]$

$P_{XYZ}(x, y, z) = P_X(x)P_Y(y)P_Z(z) \rightarrow P_{XY}(x, y) = [\sum_{z}^{P} {}_{XYZ}(x, y, z)]$
$= P_X(x)P_Y(y)[\sum_{z}^{P} {}_{Z}(z)]$
$= P_X(x)P_Y(y)$

Conditional Independence: X and Y are independent conditioned on Z if
$P_{XY|Z=z}(x, y) = P_{X|Z=z}(x)P_{Y|Z=z}(y)$ for any (x,y,z).

Example 6

X,Y,Z are all 0,1 valued RV's
1. Suppose that X and Y are independent, with
$P_X(0) = P_X(1) = P_Y(0) = P_Y(1) = \frac{1}{2}$ and suppose $Z = X + Y mod 2$
2. Suppose $P_X(0) = \frac{1}{3}, P_X(1) = \frac{2}{3}$, and $Y = X, Z = X$.

For these two cases, are X and Y independent, given Z?

Answer:

1.$No.$
2.$Yes.$