Lectures of June 30th, 2006
Scribe: Taha. S. Morbiwalla

Continuation of Poisson Process

5. Continued:
Let $Z(t)$ be the random impulse train in the form of

$z(t) = \sum_{k=1}^{\infty} \delta(t - T_k)$ - underlying a Poisson process where T_k's are time instance at which random events occur. Let $h(t) = u(t)$ be the impulse response of an LTI system. The output of the system with input $z(t)$ is the Poisson process $n(t)$, i.e.,

$$n(t) = z(t) * u(t)$$

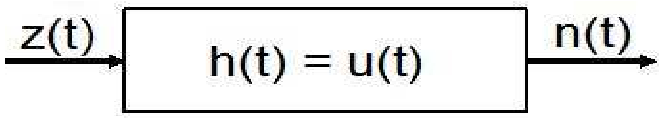

Figure 1:

6. If the impulse response of the above LTI system is not $u(t)$, but a general function, the output process is called *a short noise process*.

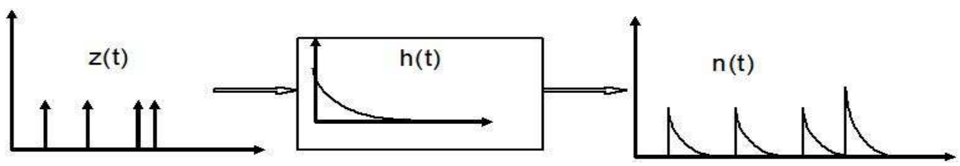

Figure 2:

For example, a photon particle when it hits a barrier.
Or in a 2D example, an earthquake.

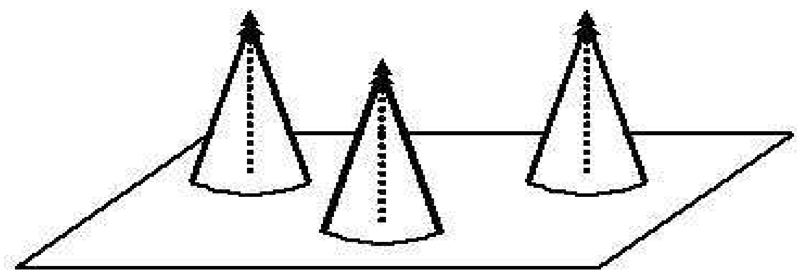

Figure 3: A 2-D example

7. The $M_N(t)$, (mean of a Poisson process) of $\alpha = \lambda t$, for a fixed t is :

$$M_N(t) = \lambda t$$

where: $N(t)$=Number of events that have happened, and $M_N(t)$ is the average number of events.

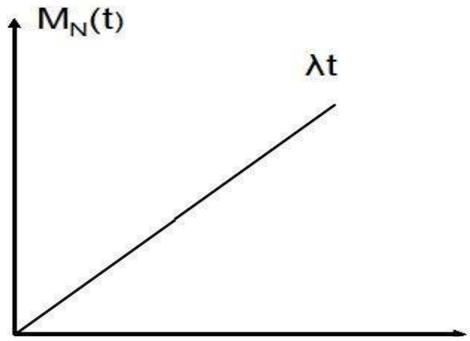

Figure 4:

The Covariance $R_N(t_1, t_2)$

Suppose $t_1 < t_2$, then
$R_N(t_1, t_2) = E[N(t_1) \cdot N(t_2)] \longrightarrow$ if we think of $N(t_2)$ as a product of other events that occurred after $N(t_1)$.

$R_N(t_1, t_2) = E[N(t_1) \cdot (N(t_1) + N(t_2 - t_1))]$
$= E[N(t_1)^2] + E[N(t_1)N(t_2 - t_1)]$
(the assumption of independence allows us to distribute the function as below.)
$= (\lambda t_1)^2 + VAR[N(t_1)] + E[N(t_1)]E[N(t_2 - t_1)]$
$= (\lambda t_1)^2 + \lambda t_1 + (\lambda t_1)(\lambda(t_2 - t_1))$
$= \lambda t_1 + \lambda^2 t_1 t_2$

In general $R_N(t_1, t_2) = \lambda min(t_1 - t_2) + \lambda^2 t_1 t_2 \longrightarrow$ the autocorrelation function.
$C_N(t_1, t_2) = R_N(t_1, t_2) - E[N(t_1)]E[N(t_2)] = \lambda min(t_1, t_2)$

This also justifies that the Poisson Process is not a W.S.S Process.

Power Spectral Density and Filtering of R.P.

Suppose process $X(t)$ is W.S.S, we will use $R_X(t)$ to denote its autocorrelation function. Specifically, t here, denotes $t_2 - t_1$, in the original notation $R_X(t_1, t_2)$ of the autocorrelation function.
The power spectral density (PSD) $S_X(t)$ of $X(t)$ is defined as
$S_X(f) = F\{R_X(t)\}$ Where F { } is either continous / discrete time

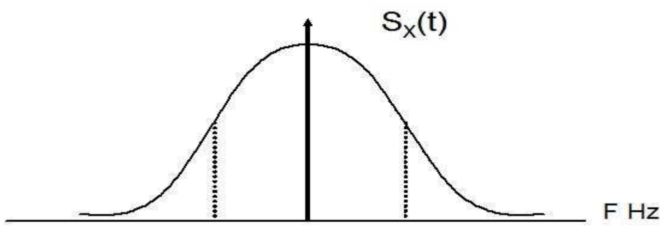

Figure 5:

It follows from the properties of Fourier Transform that
$\int_{-\infty}^{\infty} S_X(f)df = R_X(0)$

This can be proven as shown below:

$x(t) \underset{\rightarrow}{F} X(f)$

by definition:
$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}$ - thus
$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$
$x(0) = \int_{-\infty}^{\infty} X(f)e^{0}df$

Thus proving the Power Spectral Density function:
$R_x(0) = \int_{-\infty}^{\infty} S_X(f)e^{0}df$

$R_x(0) = E[X(t).X(t)]$ - For any t. *By definition of the Autocorrelation function.*

$R_x(0) = VAR[X(t)] + (E[X(t)])^2$

$R_X(0)$ means no change in t.

If we interpret the Autocorrelation function as Power, the $VAR[X(t)]$ can be thought of as the DC Power, and $(E[X(t)])^2$ as the AC power, then we can say that $R_X(0) =$ to the entire power AC + DC.

Important Note: The $S_X(f)$ is always Non-Negative.
$S_X(t)$ can be interpreted as the distribution of Power.

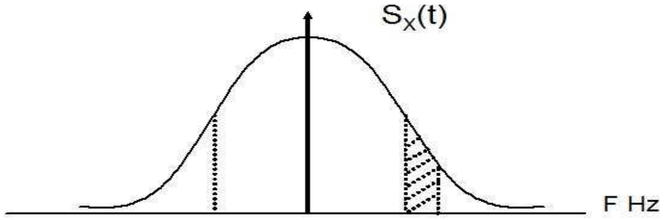

Figure 6:

Example 1.
Let $X(t)$ be a W.S.S process with $S_X(f)$ specified as follows (Fig 7.)

1. Find the total power of the process
2. Find the two sided power of the process within 10Hz and 30Hz

Sol:

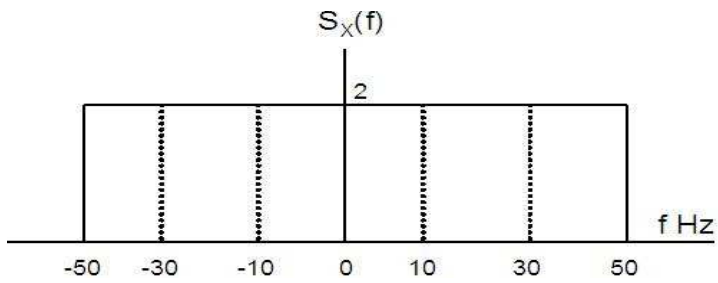

Figure 7:

1. Total Power $= \int_{-\infty}^{\infty} S_X(f)df = \int_{-50}^{50} 1df = 100$

2.Over the interval [10Hz and 30Hz]
$= \int_{10}^{30} S_X(f)df + \int_{-30}^{-10} S_X(f)df$
$= 20 + 20 = 40$W

Example 2.

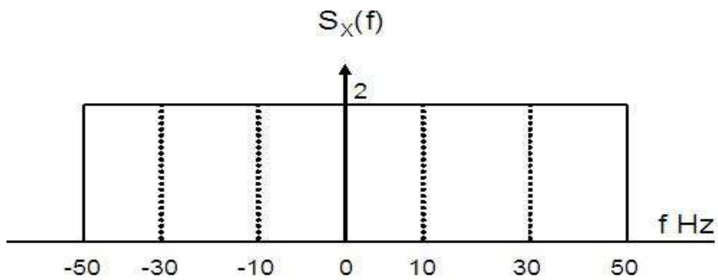

Figure 8:

1. What is the total power in the $S_x(f)$ specified below.
$\delta(t) =$ dc power.

The answer is similar to example 1, but also includes a constant of 2.

Filtering of W.S.S Random Process

Setting:

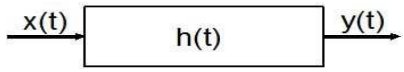

Figure 9:

$x(t)$ is W.S.S
$h(t)$ is the impulse response

Define
$R_h(t) = h(t) * h(-t)$ i.e
$R_h(t) = \int_{-\infty}^{\infty} h(\tau - t)h(\tau)d\tau \rightarrow$ Autocorrelation function

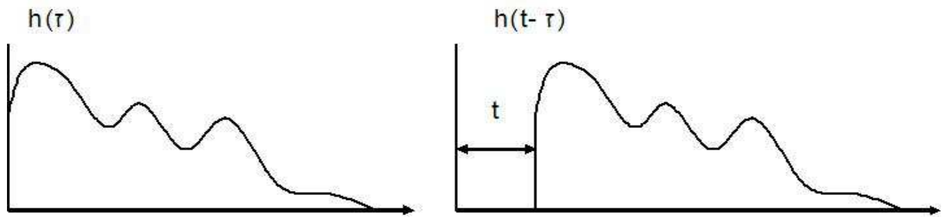

Figure 10:

$R_h(t)$ is also known as the deterministic autocorrelation function. e.g.
if the process is sinusoidal, and we shift it by 1 period - we get the convolution as shown below:

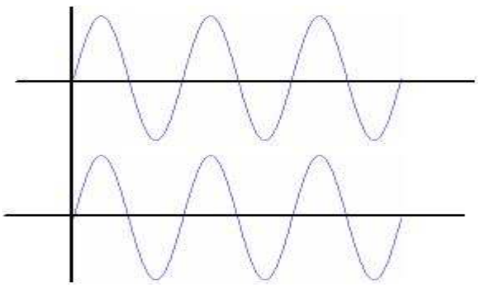

Figure 11:

However, If we shift it by $^1/_2$ a period, we get negative values.

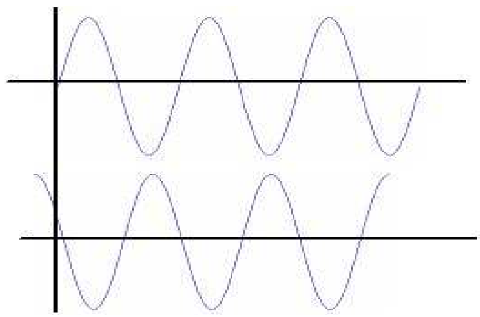

Figure 12:

Define $S_h(t) = F\{R_h(t)\} = \mid H(f) \mid^2 = H(f).H(f)$

$y(t) = x(t) * h(t)$

$$\underbrace{\mid Y(f) \mid^2}_{R_y(t)} = \underbrace{\mid X(f) \mid^2}_{R_x(t)} \cdot \underbrace{\mid H(f) \mid^2}_{R_h(t)}$$

Key Results

1. $M_y = H(0).M_x$ where $H(0) = \int_{-\infty}^{\infty} h(t)dt = H(f) \mid_{f=0}$

This simply means that the Output Mean M_y is a scaled version of the Input mean M_x. Where the scaling factor is precisely given by the area under $h(t)$.

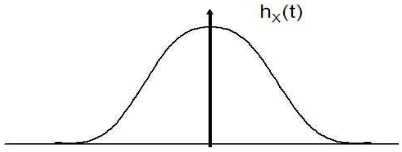

Figure 13:

2. If $x(t)$ is W.S.S, then $y(t)$ is also W.S.S.
3. $R_y(t) = R_x(t) * R_h(t)$
4. $S_y(f) = S_x(f).\underbrace{S_h(f)}_{\text{Future Scaling Factor}}$

$S_x(f) =$ The power distribution of a R.P across different bands of Input.

A process is said to be "White" if its PSD is flat, namely across all frequencies.

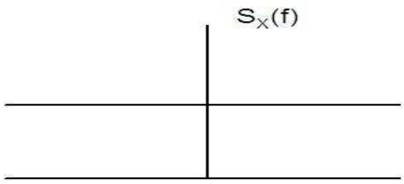

Figure 14: