Lectures of 8:00 - 10:00 a.m. June 30, 2006
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1 IID Process

IID process is a discrete time random process, $X(n)$,(or random sequence $X(1)$, $X(2)$ $\cdots\cdots$) is called an IID process, if the sequence $X(1)$, $X(2)$ $\cdots\cdots$ is IDD.

- an IID process is completely specified by the common CDF or PDF of random variable.
- mean $m_x(n)$ of IID process $X(n)$ is a constant.
- the autocorrelation function of IID process $X(n)$:
  suppose $n_1 \leqslant n_2$,
  we have:
  $R_X(n_1, n_2) = E[X(n_1)X(n_2)]$

$$= \begin{cases} E[X(n_1)] \cdot E[X(n_2)], & \text{if } n_1 < n_2 \\ E[X^2(n_1)], & n_1 = n_2 \end{cases} \tag{1}$$

$$= \begin{cases} m^2 & \text{, if } n_1 < n_2 \\ m^2 + \sigma^2 & , n_1 = n_2 \end{cases} \tag{2}$$

(where σ^2 is the variance of each $X(i)$)

In general:
$R_X(n_1, n_2) = m^2 + \sigma^2 * \delta n_1, n_2$
where $\delta n_1, n_2$ is defined as:

$$\delta n_1, n_2 = \begin{cases} 1, & n_1 = n_2, \\ 0, & n_1 \neq n_2, \end{cases} \tag{3}$$

$$\delta n_1, n_2 = \delta(n_1 - n_2) \tag{4}$$

$$C_X(n_1, n_2) = R_x(n_1, n_2) - m_x(n_1) \cdot m_x(n_2) \tag{5}$$

$$C_X(n_1, n_2) = \sigma^2 \cdot \delta n_1, n_2 \tag{6}$$

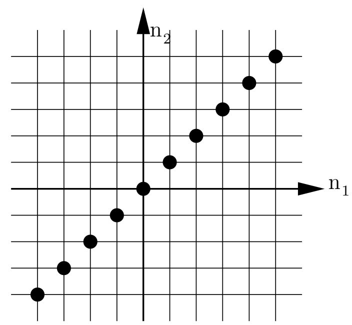

Figure 1: $C_X(n_1, n_2)$

- An IID process is W.S.S.
- An IID process is S.S.S.

Example: suppose the $X(n)$ is an IID random process, where at each n, $X(n)$is Gaussian with mean μ and variance σ^2, find the joint PDF of R.V's $(X(2), X(3), X(8))$.

Solution: $X(2), X(3), X(8)$ are jointly Gaussian

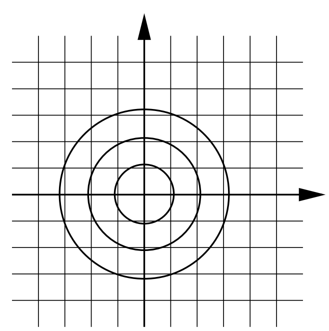

Figure 2: Jointly Gaussian distribution

Their mean vector is $\overrightarrow{m} = (\mu, \mu, \mu)$

Their covariance matrix is :

K=$\begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}$

Their joint PDF:

$$f_{X(2),X(3),X(8)}(x_2,x_3,x_8) = \frac{1}{(2\pi)^{\frac{3}{2}} \cdot |K|^{\frac{1}{2}}} \cdot e^{\left(\frac{-[(x_2,x_3,x_8)-\vec{m}]^T \cdot K[(x_2,x_3,x_8)-\vec{m}]}{2}\right)} \tag{7}$$

Alternatively, it is also clear that:

$$f_{X(2),X(3),X(8)}(x_2,x_3,x_8) = f_{X(2)}(x_2) \cdot f_{x(3)}(x_3) \cdot f_{x(8)}(x_8) \tag{8}$$

$$f_{X(2),X(3),X(8)}(x_2,x_3,x_8) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(x_2-\mu)^2}{2\sigma^2}\right)} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(x_3-\mu)^2}{2\sigma^2}\right)} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{\left(-\frac{(x_8-\mu)^2}{2\sigma^2}\right)} \tag{9}$$

2 Sum Process

Suppose $X(n)$ is an IID process, the process:

$$S(n) := \sum_{i=1}^{n} X(i) \tag{10}$$

is called a **sum process**

- $S(n)$ is completely specified by the underlying IID process $X(n)$.
-
  $$S(n) = S(n-1) + X(n) \tag{11}$$

  for any n

$$S(n) = X(1) + \ldots + X(n) \tag{12}$$
$$S(n-1) = X(1) + \ldots + X(n-1) \tag{13}$$

- $S(n)$ is Markov
- $S(n)$ is NOT W.S.S.
- $S(n)$ is NOT S.S.S.

Example: Let $S(n)$ be the sum process defined by $S(n) := \sum_{i=1}^{n} X(i)$ where IID process $X(n)$ is defined in the previous example. Find the joint PDF of $X(n_1)$ and $X(n_2)$ for any given pair (n_1, n_2).

Solution: For any n, $S(n)$ is a Gaussian random variable.

(aside:

- If $X_1 \ldots X_n$ are jointly Gaussian, then:
  $\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} := A \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ is jointly Gaussian for any linear transformation A
  (text $P.242$)
- if $X_1 \ldots X_n$ are jointly Gaussian, then $Y := X_1 + \ldots + X_n$ is Gaussian.
  (text $P.224, Ex.4.50$) aside end)

The variance of $S(n)$ is $n\sigma^2$, and the mean of $S(n)$ is $n\mu$.

If $n_1 < n_2$, $[S(1) \ldots S(n_2)]$ is a linear transform of $[X(1) \ldots X(n_2)]$

So $S(n_1)$ and $S(n_2)$ are jointly Gaussian.

$$f_{s(n_1),s(n_2)}(s_1, s_2) = f_{s(n_1)}(s_1) \cdot f_{s(n_1)|s(n_2)}(s_1|s_2) \tag{14}$$
$$f_{s(n_1),s(n_2)}(s_1, s_2) = f_{s(n_1)}(s_1) \cdot f_{s(n_1-n_2)}(s_1 - s_2) \tag{15}$$

$$f_{s(n_1),s(n_2)}(s_1, s_2) = \frac{1}{\sqrt{(2\pi\sigma^2 n_1)}} e^{(\frac{-(s_1-n_1\mu)^2}{2n_1\sigma^2})} \cdot \frac{1}{\sqrt{(2\pi\sigma^2(n_2-n_1))}} e^{(\frac{-(s_2-s_1-(n_2-n_1)\mu)^2}{2(n_2-n_1)\sigma^2})} \tag{16}$$

Remark:

A continuous/discrete time process $X(t)$ is called a Gaussian process if for any k, and any $t_1, t_2 \ldots t_k$, $X(t_1), X(t_2) \ldots X(t_k)$ are jointly Gaussian.

3 Poisson Process

- Continuous-time process
- Setting:
  1. Random events occur at an average rate of λ per time unit

2. When the time axis is divided into sufficiently small intervals such that at most one event can occur in any given interval
    (a) The probability that an event occurs in a given interval is the same across all intervals.
    (b) Whether an event occurs in an interval is independent across intervals.

Let $N(t)$ be the number of events occuring during time window $[0, t]$.

When $N(t)$ is treated as a (random) function of t, the ensemble of such function is called a Poisson process.

- A Poisson process is completely specified by parameter λ

Remarks:

1. $N(t)$ at any t is a Poisson R.V. with parameter $\alpha = \lambda t$
   i.e. the PMF of $N(t)$ is

$$P\left[N(t) = k\right] = \frac{(\lambda\, t)^k}{k!} e^{-\lambda\, t} \tag{17}$$

2. For any given time (t_1, t_2) with $t_1 < t_2$,

$$P\left[N(t_1) = i, N(t_2) = j\right] = P\left[N(t_1) = i\right] \cdot P\left[N(t_2) = j | N(t_1) = i\right] \tag{18}$$

$$P\left[N(t_1) = i, N(t_2) = j\right] = P\left[N(t_1) = i\right] \cdot P\left[N(t_2 - t_1) = j - i\right] \tag{19}$$

$$P\left[N(t_1) = i, N(t_2) = j\right] = \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \cdot \frac{(\lambda(t_2 - t_1))^{j-i} e^{-\lambda(t_2 - t_1)}}{(j - i)!} \tag{20}$$

3. $N(t)$ is Markov
4. $N(t)$ is NOT W.S.S.
5. Every realization of $N(t)$ can be treated as the integral of a “randomly-spaced impulse train”

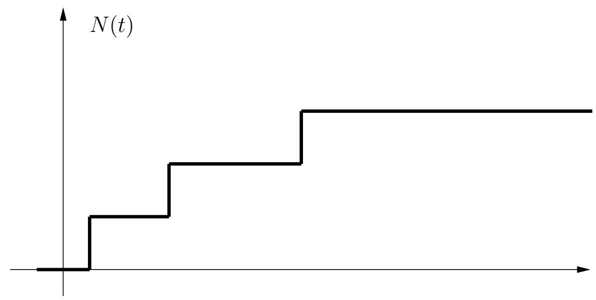

Figure 3: Integral of a randomly-spaced impulse train

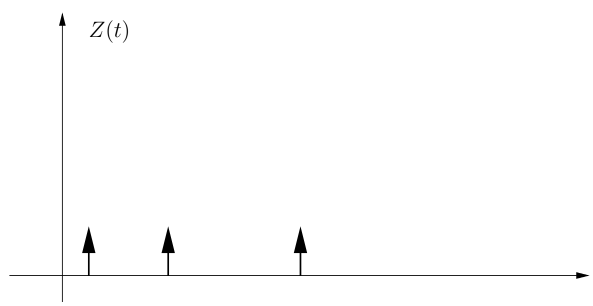

Figure 4: A randomly-spaced impulse train

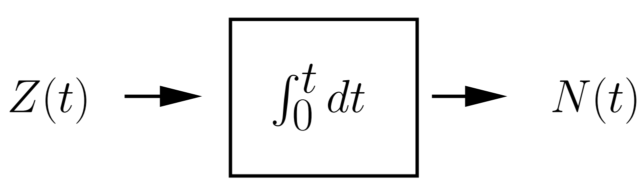

Figure 5: Filter with impulse response $h(t) = u(t)$