Lectures of June 27th, 2006
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1 Markov Process

1.1 Definition of Markov Process
A Random Process, $X(t)$, is said to be Markov if for any positive integer, $k \leq 3$, any choice of k time instants $t_1, t_2, \ldots, t_k$ with $t_1 < t_2 < \ldots < t_k$, $X(t_k)$ is independent of $(X(t_1), X(t_2), \ldots, X(t_{k-2}))$ conditioned on $X(t_{k-1})$.

Aside: Conditional Independence
For any three Random Variables X,Y, and Z, X is said to be independent of Z given Y if:
$f_{X|YZ}(x|y,z) = f_{X|Y}(x|y)$ *in the Continuous setting or*
$P_{X|YZ}(x|y,z) = P_{X|Y}(x|y)$ *in the Discrete setting.*

A Markov Chain has an arbitrary number of variables.

$$X(t_1) - X(t_2) - \ldots - X(t_{k-1}) - X(t_k)$$

Therefore, a process is Markov, if for any $k \geq 3$, and any $t_1 < t_2 < \ldots < t_k$:

$$f_{X(t_k)|X(t_{k-1}),\ldots,X(t_1)}(x_k|x_{k-1},\ldots,x_1) = f_{X(t_k)|X(t_{k-1})}(x_k|x_{k-1})$$

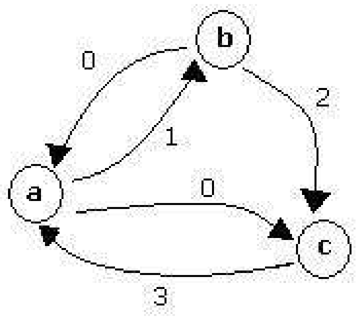
EXAMPLE 1)
Consider a counter, C(t), where t is the time instance and consider the counter at t=4,5,6. Probability distribution of C(6) does not depend on C(4) if C(5) is given. Therefore, to compute C(6), we do not need to know anything about C(4).

$$f_{C(6)|C(5),C(4)}(C(6)|C(5),C(4)) = f_{C(6)|C(5)}(C(6)|C(5))$$

1.2 Markov Process exists in all engineering science
This is shown in Examples 2 to 5.

EXAMPLE 2) Weather prediction.
There is no need to look at the past weather since everything is summed up in today's weather. Just using today's weather, it is possible to predict future weather. The future is independent of the past given today.

EXAMPLE 3) ELG1100 state transition diagram
Figure 1.1



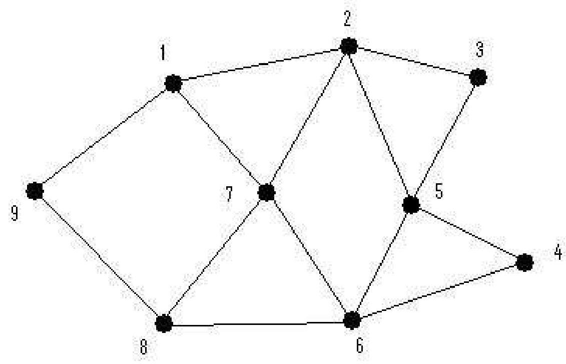
State Transition Diagram

Given the current state, the next state can be predicted without knowledge of the previous state.

EXAMPLE 4)
In the counter process in the previous example, the process is Markov. Given C(1),C(3),C(5),C(10)

If C(5) is 5, you don't need to know C(1) or C(3) to predict C(10) in terms of probability.

EXAMPLE 5)
*Please note that this example is for extra knowledge and students do not need to be responsible for this example.
Figure 1.2



Perron Frobenius "bug crawling" Theorem

It is given that there exists a path from a node to any other node and that there is a bug on this node diagram. It does not matter where the bug is at $t = 0$. As $t \to \infty$ the probability distribution stays the same.

An application of this concept is first version of the Google Search Engine.

2 Mean, Auto-Correlation and Auto-Covariance function

2.1 Definitions

Given a Random Process, $X(t)$, its mean $M_x(t)$ is a function of t, defined by:

$$m_x(t) = E[X(t)]$$

Its auto-correlation function $R_x(t_1, t_2)$ is a two-variable function defined as:

$$R_x(t_1, t_2) = E[X(t_1).X(t_2)]$$

Its auto-covariance function $C_x(t_1, t_2)$ is a two-variable function defined by

$$C_x(t_1, t_2) = E[(X(t_1) - m_x(t_1)).(X(t_2) - m_x(t_2))]$$

REMARKS

$\rightarrow R_x(t_2, t_1) = R_x(t_1, t_2)$
$\rightarrow C_x(t_2, t_1) = C_x(t_1, t_2)$

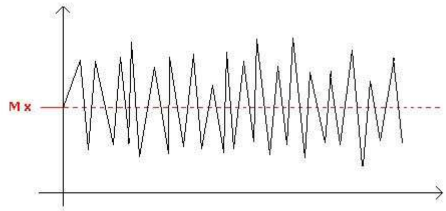
$$\begin{aligned}
\rightarrow C_x(t_1, t_2) &= E[(X(t_1) - m_x(t_1)).(X(t_2) - m_x(t_2)) \\
&= E[X(t_1)X(t_2) - m_x(t_1)X(t_2) - X(t_1)m_x(t_2) + m_x(t_1)m_x(t_2) \\
&= E[X(t_1)X(t_2)] - m_x(t_1)E[X(t_2)] - m_x(t_2)E[X(t_1)] + m_x(t_1)m_x(t_2) \\
&= E[X(t_1)X(t_2)] - m_x(t_1)m_x(t_2) - m_x(t_2)m_x(t_1) + m_x(t_1)m_x(t_2) \\
&= R_x(t_1, t_2) - m_x(t_1)m_x(t_2)
\end{aligned}$$

$$\begin{aligned}
\rightarrow C_x(t_1, t_2) &= E[X^2(t_1)] - (E[X(t_1)])^2 \\
&= VAR[X(t_1)]
\end{aligned}$$

2.2 Intuitive meaning in Electrical Engineering

Given an arbitrary signal like in the figure below:

Figure 2.1



We as Electrical Engineers should understand that:

→ $E[X] = M_x$ is the average value of the signal, also known as the DC component.

→ $E[X^2]$,the mean-squared value, is defined as the average power of the signal.

→ $E[(X - M_x)^2] = VAR[X]$ is defined as the power of the signal with its mean removed, which means that VAR[X] gives the AC component of the signal.

EXAMPLE 6)

Given $X(t) = A\cos(2\pi t)$, where A is a uniform over [0,1]. Find its auto-correlation function and the auto-covariance function.

SOLUTION

$$\begin{aligned}
R_x(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= E[A\cos(2\pi t_1).A\cos(2\pi t_2)] \\
&= E[A^2\cos(2\pi t_1)\cos(2\pi t_2)] \\
&= \cos(2\pi t_1)\cos(2\pi t_2)E[A^2] \\
&= \cos(2\pi t_1)\cos(2\pi t_2)\int_0^1 a^2\,da \\
&= \frac{1}{3}\cos(2\pi t_1)\cos(2\pi t_2)
\end{aligned}$$

$$\begin{aligned}
C_x(t_1, t_2) &= R_x(t_1, t_2) - m_x(t_1)m_x(t_2) \\
&= \frac{1}{3}\cos(2\pi t_1)\cos(2\pi t_2) - (\frac{1}{2}\cos(2\pi t_1))(\frac{1}{2}\cos(2\pi t_2)) \\
&= \frac{1}{12}\cos(2\pi t_1)\cos(2\pi t_2)
\end{aligned}$$

EXAMPLE 7)
Given $X(t) = \cos(2\pi t + \phi)$, where ϕ is uniform over $[0, 2\pi)$. Find $R_x(t_1, t_2)$

SOLUTION

$$\begin{aligned} R_x(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= E[\cos(2\pi t_1 + \phi).\cos(2\pi t_2 + \phi)] ---------- (@) \\ &= \frac{1}{2}E[\cos(2\pi(t_1+t_2)+2\phi) + \cos(2\pi(t_1-t_2))] \\ &= \frac{1}{2}\cos(2\pi(t_1-t_2)) + \frac{1}{2}E[\cos(2\pi(t_1+t_2)+2\phi)] --- (1) \end{aligned}$$

where the Trigonometric Identity: $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$ is used in line (@).

Since,
$\rightarrow$ 2ϕ is uniform over $[0, 4\pi)$
$\rightarrow$ 2ϕ mod 2π is uniform over $[0, 2\pi)$
$\rightarrow$ $\cos(\alpha + 2\phi) = \cos(\alpha + (2\phi \; mod \; 2\pi)) = \cos(z)$ where z is uniform over $[0, 2\pi)$
$\rightarrow$ Therefore, $E[\cos(2\pi(t_1+t_2)+\phi)] = 0$

Therefore from (1),

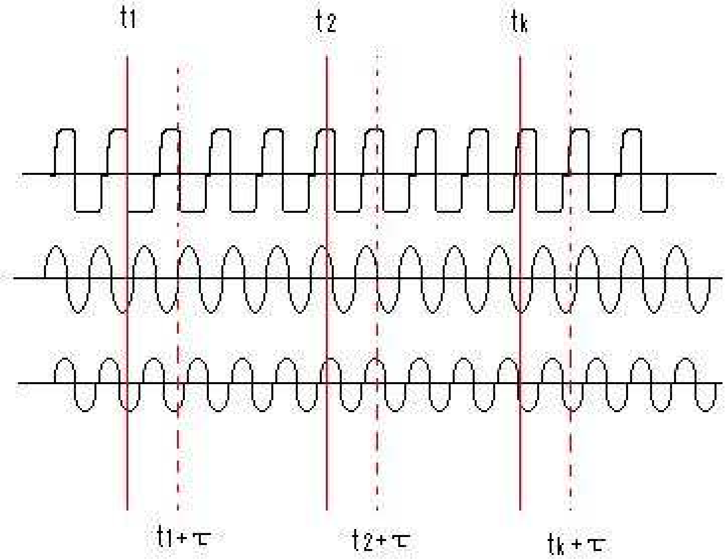
$$R_x(t_1,t_2) = \frac{1}{2}\cos(2\pi(t_1-t_2))$$

3 Strict-Sense And Wide-Sense Stationary

3.1 Stationary (or Strict-Sense Stationary) Process
A Random Process $X(t)$ is said to be stationary or Strict-Sense Stationary (SSS) if for any positive integer k and choice of k time instants $t_1, \ldots, t_k$ and τ :

$$F_{x(t_1),x(t_2),\ldots x(t_k)}(x_1, x_2, \ldots x_k) = F_{X(t_1+\tau)X(t_2+\tau)\ldots X(t_k+\tau)}(x_1, x_2, \ldots, x_k)$$

Figure 3.1: *Note that the values at t_k is equal to values at $t_{k+\tau}$.*



3.2 Wide-Sense Stationary (WSS) Process
A Random Process $X(t)$ is said to be Wide-Sense Stationary (WSS) if

1) $M_x(t) = m$ for some constant m
2) $R_x(t, t+\tau) = r(\tau)$ for some function $r(.)$

Remember: $R_x(t_1, t_2) = \frac{1}{2}\cos(2\pi(t_1 - t_2))$

REMARKS

→ If a process $X(t)$ is W.S.S then $R_x(t_1, t_2)$ is also written as a unit-variance function $R_x(\tau)$.

→ In this case (WSS):

$\diamond R_x(\tau) = R_x(-\tau)$

$\diamond R_x(t_1, t_2) = R_x(t_2, t_1)$

→ If $X(t)$ is SSS, then it is WSS.

From EXAMPLE 7: $X(t) = \cos(2\pi t + \phi)$ where ϕ is uniform over $[0, 2\pi)$

(1) Is $X(t)$ WSS?
(2) Is $X(t)$ SSS?

SOLUTION
Since

(1)

$$\left\{ \begin{array}{l} m_x(t) = 0 \\ R_x(t_1, t_2) = \frac{1}{2}\cos(2\pi(t_1 - t_2)) \end{array} \right\} \Rightarrow X(t) \; is \; WSS.$$

(2) Since it is WSS, it must be SSS.