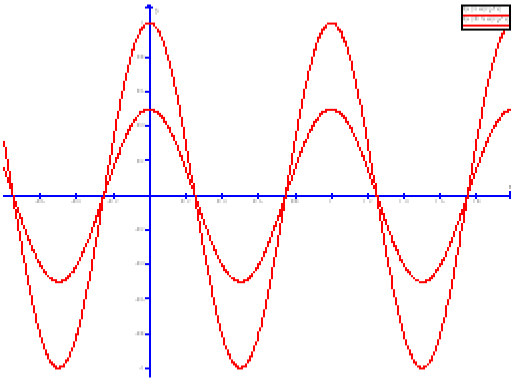
Lecture of June 23rd, 2006
Scribe: David Beauchamp, Ilir Zymberi, Anis Shaklab

Example 1 *A Random process (RV) X(t) is defined as:* $X(t) = A\cos(2\pi t)$ *where* $t\epsilon(-\infty \text{ to } \infty)$
Where A is a Random Variable uniformly distributed over [0,1)

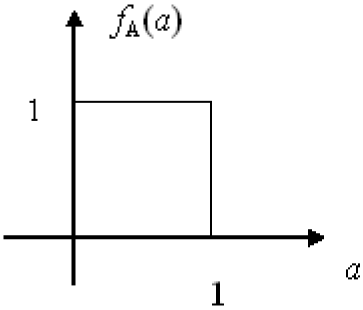
1. *Find* $E[X(t)]$ *as a function of t*
2. *Find the PDF of X(t)*



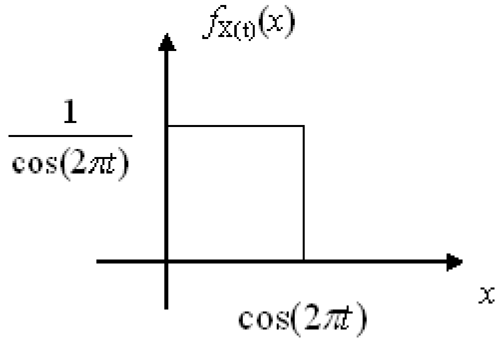
Solution:

1. For a fixed t;

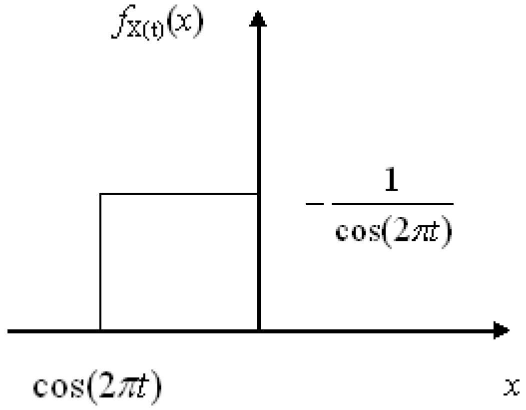
$$\begin{aligned} E[x(t)] &= E[A\cos(2\pi t)] \\ &= E[A]\cos 2\pi t \\ &= \tfrac{1}{2}\cos(2\pi t) \end{aligned}$$



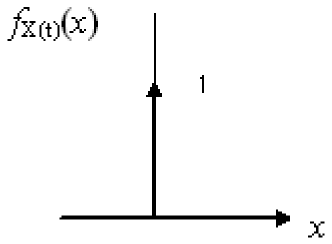
2. For a given t, RV $X(t) = A\cos(2\pi t)$, which is simply a scaled version of A
   If $\cos(2\pi t)\epsilon 0$, x(t) is uniform over [o,cos$(2\pi t)$)



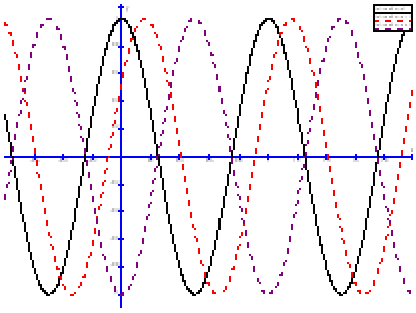
If $\cos(2\pi t) < 0$, x(t) is uniform over $(\cos(2\pi t),0]$



If $\cos(2\pi t) = 0$



Example 2 *A Random Process (RP)* $X(t) = A\cos(2\pi t + \theta)$ *where* θ *is uniformly distributed over* $[0, 2\pi)$*. Find all the PDFs of X(t).*



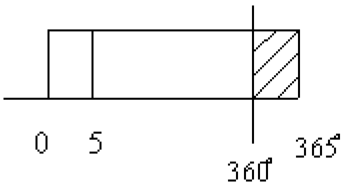
Solution:

For any given t;

$X(t) = \cos([2\pi t + \theta] mod 2\pi)$

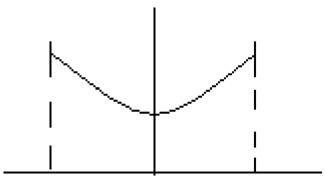
Let $Z(t) = (2\pi t + \theta) mod 2\pi$

Since θ *is uniformly distributed over* $[0, 2\pi)$, $Z(t)$ *for any given* t *is uniform over* $[0, 2\pi)$



$X(t) = \cos(Z(t))$, *where* $Z(t)$ *is uniform over* $[0, 2\pi)$

Recall from a previous example that the PDF of $Y = \cos$ *for* X *uniform over* $[0, 2\pi)$ *is* $\frac{1}{\sqrt{x-y^2}\pi}$



Therefore, the PDF of X(t) is $f_{\mathrm{x(t)}}=\frac{1}{\sqrt{x-y^2}\pi}$. We note the PDF of X(t) is independent of t.

Example 3 *A counter is set to zero at time n=0. We will keep tossing a coin at every discrete-time instant* $n > 0$.
If we see "H", we increase the Counter by 1.

If we see "T", we decrease the counter by1.
We assume that the coin is biased where $P[H] = 1/3$
We will denote the reading of counter at time n by C(n).

1. *Find $E[C(n)]$ for every $n \geq 0$*
2. *Find the PMF of C(3)*

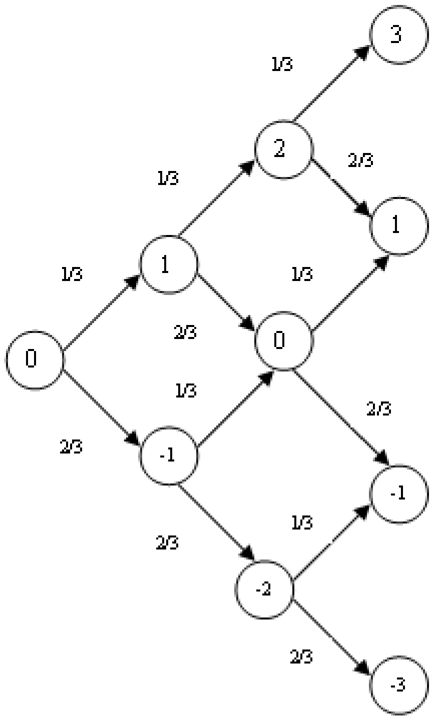
Solution:

1. Let

$$Z(n) = \begin{cases} 1, & If\ seeing\ H\ at\ time\ n, \\ -1, & If\ seeing\ H\ at\ time\ -n. \end{cases}$$

   $C(n) = C(n-1) + Z(n)$
   $C(0) = 0$
   $C(n) = C(o) + Z(1) + Z(2) + ... + Z(n)$
   $E[C(n)] = E[C(0)] + \sum_{i=1}^{n} E[Z(i)]$

   $E[Z_i] = 1\frac{1}{3} + (-1)\frac{2}{3} = \frac{-1}{3}$
   $E[C(n)] = \frac{-1}{3}n$



2.

$P_{C(3)}(3) = (\frac{1}{3})^3$
$P_{C(3)}(1) = (\frac{1}{3})^2 \frac{2}{3} + (\frac{1}{3})^2 \frac{2}{3} + (\frac{1}{3})^2 \frac{2}{3}$
$P_{C(3)}(-1) = 3(\frac{1}{3})^2 \frac{2}{3}$
$P_{C(3)}(-3) = (\frac{2}{3})^3$

$$\begin{bmatrix} P_{\mathrm{C(1)}}(1) \\ P_{\mathrm{C(1)}}(-1) \end{bmatrix} = \begin{bmatrix} P[C(1)=1] * P[C(0)=0] \\ P[C(1)=-1] * P[C(0)=0] \end{bmatrix}$$

$$= \begin{bmatrix} P[C(0)=0] \end{bmatrix} * \begin{bmatrix} P[C(1)=1]|C(0)=0] \\ P[C(1)=-1]|C(0)=0] \end{bmatrix} \begin{bmatrix} P[C(2)=2] \\ P[C(2)=0] \\ P[C(2)=-2] \end{bmatrix}$$

$$= \begin{bmatrix} P[C(2)=2|C(1)=1] & P[C(2)=2|C(1)=-1] \\ P[C(2)=0|C(1)=1] & P[C(2)=0|C(1)=-1] \\ P[C(2)=-2|C(1)=1] & P[C(2)=-2|C(1)=-1] \end{bmatrix} * \begin{bmatrix} P[C(1)=1] \\ P[C(1)=-1] \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 0 \\ 2/3 & 1/3] \\ 0 & 2/3] \end{bmatrix} \begin{bmatrix} P[C(1)=1] \\ P[C(1)=-1] \end{bmatrix}$$

$$\begin{bmatrix} P[C(3)=3] \\ P[C(3)=1] \\ P[C(3)=-1] \\ P[C(3)=-3] \end{bmatrix}$$

$$= \begin{bmatrix} P_{\mathrm{C(3)|C(2)}}(3|2) & P_{\mathrm{C(3)|C(2)}}(3|0) & P_{\mathrm{C(3)|C(2)}}(3|-2) \\ P_{\mathrm{C(3)|C(2)}}(1|2) & P_{\mathrm{C(3)|C(2)}}(1|0) & P_{\mathrm{C(3)|C(2)}}(1|-2) \\ P_{\mathrm{C(3)|C(2)}}(-1|2) & P_{\mathrm{C(3)|C(2)}}(-1|0) & P_{\mathrm{C(3)|C(2)}}(-1|-2) \\ P_{\mathrm{C(3)|C(2)}}(-3|2) & P_{\mathrm{C(3)|C(2)}}(-3|0) & P_{\mathrm{C(3)|C(2)}}(-3|-2) \end{bmatrix} * \begin{bmatrix} P[C(2)=2] \\ P[C(2)=0] \\ P[C(2)=-1] \end{bmatrix}$$