Lecture of June 23rd, 2006 - 8:00am
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1 The Central Limit Theorem

Theorem [Central Limit Theorem] Let $X_1, X_2, \ldots X_n$ be iid $RV's$ with mean μ and variance σ^2. Define $RV Z_n$ by

$$Z_n := \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sqrt{n}\sigma}$$

Then $n \overset{lim}{\rightarrow} \infty \ P[Z_n \leq z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ (i.e. Gaussian CDF with zero mean and unit-variance).

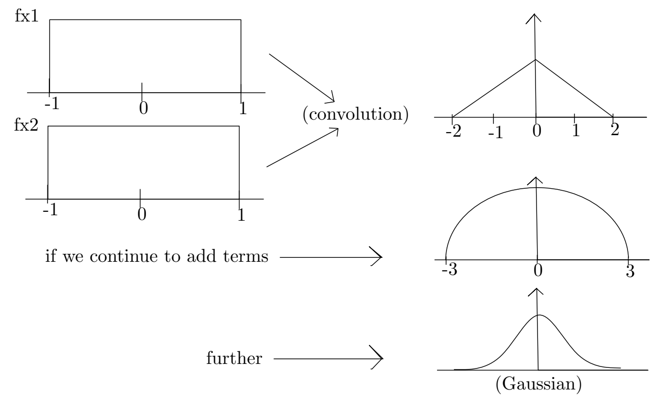
Remarks

1. $E[X+Y] = E[X] + E[Y]$
2. $VAR[X+Y] = E[(X+Y-\mu_{X+Y})^2]$

$$\begin{aligned} &= E[X^2 + Y^2 + {\mu_{X+Y}}^2 + 2XY - 2X\mu_{X+Y} - 2Y_{X+Y}] \\ &= E[X^2 + Y^2 + (\mu_X + \mu_Y)^2 + 2XY - 2X\mu_X - 2Y\mu_Y] \\ &= E[(X-\mu_X)^2 + (Y-\mu_Y)^2 - 2(X-\mu_X)(Y-\mu_Y)] \\ &= VAR[X] + VAR[Y] - 2COV[X,Y]. \end{aligned}$$

   When $X+Y$ are independent, $X+Y$ are uncorrelated, i.e. $COV[X,Y] = 0$
   That is, if $X+Y$ are independent, $VAR[X+Y] = VAR[X] + VAR[Y]$.

3. $E[\alpha X] = \alpha E[X]$
   $VAR[\alpha X] = \alpha^2 VAR[X]$
4. From 1), 2) & 3) we see that Z_n has mean zero and variance 1 (Distribution actually looks like a Gaussian).
5. The C.L.T. suggests that the distribution of Z_n is "close" to Gaussian for large n.



2 Random Processes

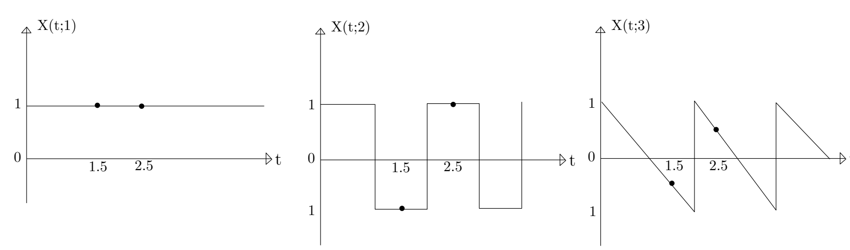
Let S be a sample space of a random experiment on which a probability law $P = [\cdot]$ is assigned. Suppose that associated with every ξ in S, there is a function of time $X_\xi(t)$, where t can be discrete time or continuous time. We may also write $X_\xi(t)$ as $X(t;\xi)$ or $X(t,\xi)$.

This family of time functions indexed by S together with the probability law on S is called a random process, or a stocastic process.

This family is called an "ensemble". Every time function in the ensemble is called a realization, a sample function or a sample path of the random process.

Example 1 $S = \{1,2,3\}$, $P[\{1\}] = P[\{2\}] = \frac{1}{4}$ *&* $P[\{3\}] = \frac{1}{2}$

The ensemble consists of the following functions,



This specifies *R.P.* $X(t)$.
[Note: we often suppress the meaning of ξ in the notation of a R.P.]

Two views of a *R.P.*

1. For any fixed $\xi \epsilon S$, $X(t;\xi)$ is a function of time.

2. For any fixed time t_o, $X(t_o)$ is a $R.V.$, mapping ξ to $X(t;\xi)$.

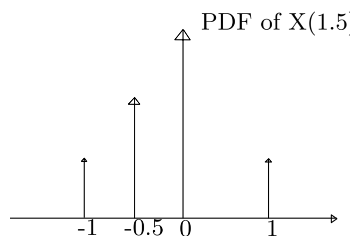
Example 2 *In **Example 1***

1. *Find* $P[X(1.5) > 0]$.
2. *Find the* PDF *of RV* $X(1.5)$.
3. *Find the joint* PDF *of* $X(1.5)$ *&* $X(2.5)$.
4. *Are* $X(1.5)$ *&* $X(2.5)$ *independent?*

Sol:

1. $P[X(1.5) > 0] = \frac{1}{4}$
2.

$$P[X(1.5) = a] = \begin{cases} \frac{1}{4}, & \text{if } a = 1, \\ \frac{1}{4}, & \text{if } a = -1, \\ \frac{1}{2}, & \text{if } a = -0.5 \\ 0, & \text{otherwise.} \end{cases}$$

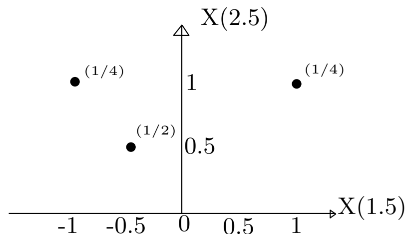


3. $(X(1.5), X(2.5))$ can only take the following configurations,

$$\begin{array}{ll} (1,1) & [\text{with probability } \frac{1}{4}] \\ (-1,1) & [\text{with probability } \frac{1}{4}] \\ (-0.5, 0.5) & [\text{with probability } \frac{1}{2}] \end{array}$$

The joint PDF of $X(1.5)$ & $X(2.5)$ is,

$$f_{X(1.5),X(2.5)}(a,b) = \frac{1}{4}\delta(a-1, b-1) + \frac{1}{4}\delta(a+1, b-1) + \frac{1}{2}\delta(a+0.5, b-0.5)$$



4. They are <u>not</u> independent.

Example 3 *R.P. $X(t)$ is defined as,*

$$X(t) = A\cos 2\pi t, t\epsilon(-\infty, +\infty)$$

where A is a RV uniformly distributed over $[0, 1)$

1. *Find $E[X(t)]$ as a function of t.*
2. *Find PDF of $X(t)$.*