1 Function of Several Random Variables

Let $X_1, X_2 \ldots X_n$ be a given set of random variables and suppose another random variable Z is defined via $X_1, X_2 \ldots X_n$ by

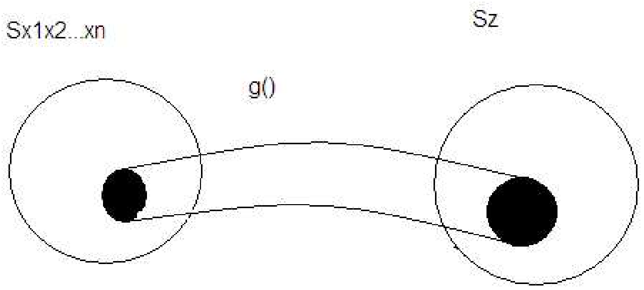
$Z := g(X_1, X_2 \ldots X_n)$

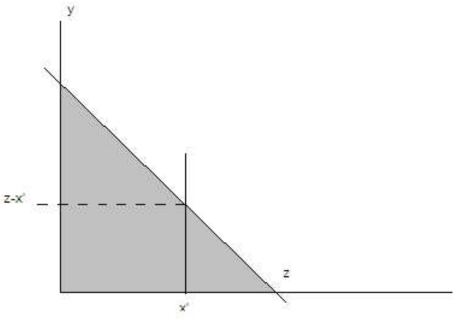
We are interested in determining the distribution of $(X_1 \ldots X_n)$ and function g.

First Principle

For any $B \subseteq S_Z$,

$P[Z \in]B = P[(X_1, X_2 \ldots X_n) \in g1(B)]$


Sx1x2...xn
Sz
g()



2 Real-Valued Functions 'g'

Summation of independent random variables

Let X and Y be independent random variables (RV's) with PDF f_x and f_y respectively. Suppose $Z = X + Y$. Find f_z in terms of f_x and f_y.

We will determine f_z via F_Z:

$F_Z(z) = P[Z \leq z]$

$= P[X + Y \leq z]$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x'} f_{xy}(x', y')dx'$

Let $h(z, x') = \int_{-\infty}^{z-x'} f_{xy}(x', y')dx'$

$F_z(z) = \int_{-\infty}^{\infty} f_{xy}(x', y')dx'$

$f_z'(z) = \frac{dF_Z(z)}{dz}$

$= \frac{d}{dz} \int_{-\infty}^{\infty} h(z, x')dx'$

$= \int_{-\infty}^{\infty} \frac{d}{dz} h(z, x')dx'$

$= \frac{d}{dz} \int_{-\infty}^{z-x'} f_{xy}(x', y')dy'$

$= f_{xy}(x', z - x')$

then $f_Z(z) = \int_{-\infty}^{\infty} f_{xy}(x', z - x')dx'$

Note: the above equation is true whether or not X and Y are independent.

Since X and Y are independent, $f_{xy}(x, y) = f_x(x)f_y(y)$

$f_Z(z) = \int_{-\infty}^{\infty} f_x(x')f_y(z - x')dx' = f_x(z) * f_y(z)$

In general, if $z = x_1 + x_2 + \ldots + x_n$, where $x_1, x_2, \ldots, x_n$ are independent, then:

$f_z(z) = f_{x_1}(z) * f_{x_2}(z) * \ldots f_{x_n}(z)$

Another more intuitive but less rigorous method to see $f_{x+y}(z) = f_x(z) * f_y(z)$ for independent X and Y is the following:

$$f_z(z) = \int_{x+y=z} f_{xy}(x, y)dxdy = \int_{-\infty}^{\infty} f_x(x)f_y(z - x)dx = f_x(z) * f_y(z)$$

In fact, such results hold for continuous, discrete, and even mixed-type RV's. The proof using characteristic funtions will show this.

$\Phi(\omega) = E[e^{j\omega Z}] = E[e^{j\omega(X+Y)}] = E[e^{j\omega X} \bullet e^{j\omega Y}]$

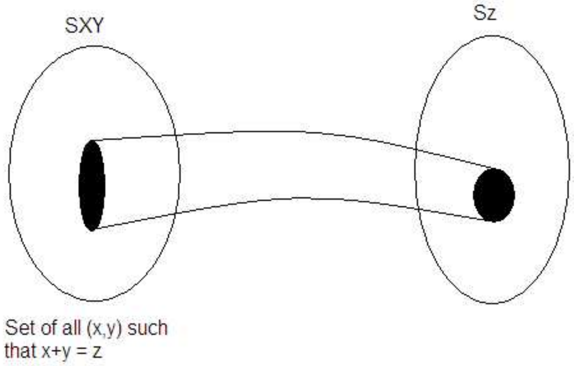
Aside: if X and Y are independent, then for any two functions g and h, $g(X)$ and $h(Y)$ are independent.

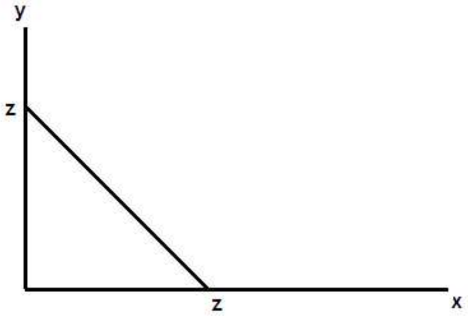
Then $e^{j\omega X}$ is independent of $e^{j\omega Y}$

Aside: if X and Y are independent, then $E[XY] = E[X]E[Y]$

Then $\Phi_2(\omega) = E[e^{j\omega X}]E[e^{j\omega Y}] = \Phi_x(\omega) \bullet \Phi_y(\omega)$

Thus, $f_2(y) = f_x(z) * f_y(z)$


SXY
Sz
Set of all (x,y) such
that x+y = z


y
z
z
x

Example

Suppose that independent messages are arriving at a station where the inter-arrival duration is modelled as an exponential distribution with parameter λ, (ie, λ is the average number of messages arriving per unit time). Suppose now is the time 0; Let Z be the time at which the fifth message will arrive. Find the PDF of Z.

Solution

(I) Let x_i be the time interval between message i and message $i-1$. (Therefore, x_1 is the time of the first message.)
Then $z = x_1 + x_2 + \ldots + x_5$

$f_z(z) = f_{x_1}(z) * \ldots f_{x_5}(z)$

Since $x_1, x_2, \ldots x_5$ are independent, we will use $\hat{f}$ to denote the fourier transform of f.

$\hat{f}(\omega) = \hat{f}_{x_1}(\omega) \bullet \ldots \hat{f}_{x_5}(\omega)$

Since

$\hat{f}_{x_1}(\omega) = \hat{f}_{x_2}(\omega) = \ldots \hat{f}_{x_5}(\omega) = F[\lambda e^{\lambda t}]$

Aside

$$e^{-\lambda t}u(t) \overset{F}{\leftrightarrow} \frac{1}{(\lambda + j\omega)}$$

$$\frac{t^{(n-1)}}{(n-1)!} \overset{F}{\leftrightarrow} \frac{1}{(\lambda + j\omega)^n}$$

Then

$$\hat{f_Z}(\omega) = (\frac{\lambda}{\lambda + j\omega})^5 = \frac{\lambda^5}{(\lambda + j\omega)^5}$$

Inverse Fourier Transform

$$\hat{f_Z}(\omega) = \frac{\lambda^5 z^4}{4!} e^{-\lambda z} u(z)$$

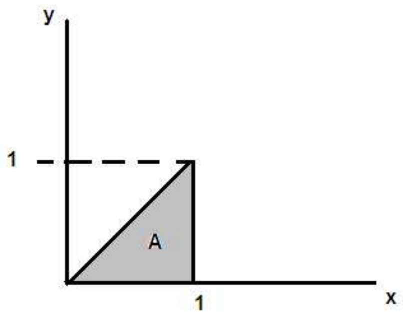
This is called m-Erlang distribution with m=5

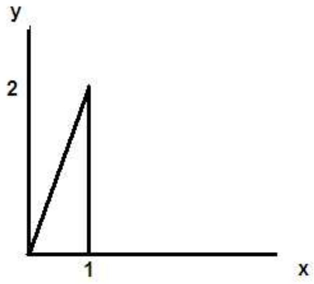
(II) Finding PDF of function of several random variable's via conditional PDF.

Example

Let(x,y) be uniformly distributed over region A.

Let $Z := XY$, Find PDF of Z


y
1
A
1
x



Solution

$$f_X(X) = \begin{cases} 2x, & \text{x} \in [0,1] \\ 0, & otherwise \end{cases}$$

We will use the fact that

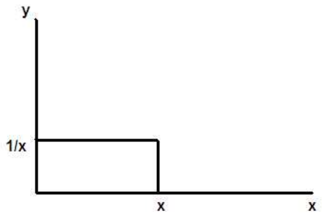
$f_Z(z) = \int_{-\infty}^{\infty} f_{Z|X=x}(z|x) \bullet f_X(x)dx$

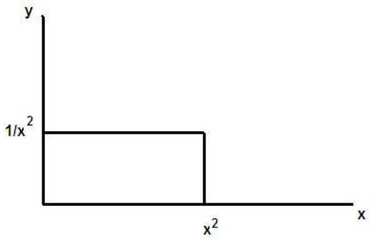
$f_{Z|X}(z|x) = f_{Z|X=x}(z)$

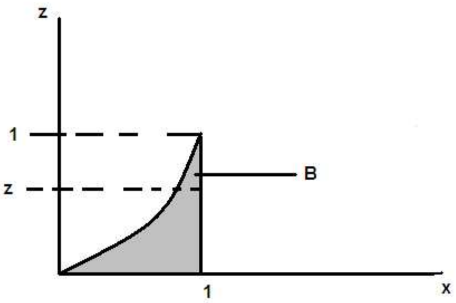
note given $X = x$, then $Z = xY$, where Y follows the distribution $f_{Y|X=x}(y)$

$$f_{Y|X=x}(y) = \begin{cases} \frac{1}{x} & \text{x} \in [0,x] \\ 0, & otherwise \end{cases}$$

As Z is now simply a scaled version of Y the distribution of Z (conditioned on $X = x$)


y
1/x
x
x


y
1/x^2
x^2
x



$$f_{Z|X=x}(z) = \begin{cases} \frac{1}{x^2} & \text{z} \in [0, x^2] \\ 0, & otherwise \end{cases}$$

$f_{zx}(z, x) = f_{z|x}(z|x) f_x(x)$

$$f_{zx}(z, x) = \begin{cases} \frac{2x}{x^2} = \frac{2}{x}, & \text{x} \in [0, 1], z \in [0, x^2] \\ 0, & otherwise \end{cases}$$

$f_z(z) = \int_{-\infty}^{\infty} f_{zx}(z, x) dx$

$= \int_{\sqrt{z}}^{1} \frac{2}{x} dx$

$= 2ln(x)|_{\sqrt{z}}^{1}$

$= -2ln\sqrt{z}$

$= -lnz for z \in [0, 1]$