Random Variables $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$ are independent iff:

$$f_{\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_n}(x_1, x_2, ..., x_n) = f_{\mathbf{X}_1}(x_1) f_{\mathbf{X}_2}(x_2) ... f_{\mathbf{X}_n}(x_n) \qquad (1)$$

$$F_{\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_n}(x_1, x_2, ..., x_n) = F_{\mathbf{X}_1}(x_1) F_{\mathbf{X}_2}(x_2) ... F_{\mathbf{X}_n}(x_n) \qquad (2)$$

To find the CDF of $\mathbf{X}$:

$$F_{\mathbf{X}_1}(x_1) = \int_{x_n} ... \int_{x_2} \int_{x_1} f_{\mathbf{X}_1,\mathbf{X}_2,...,\mathbf{X}_n}(x_1, x_2, ..., x_n) dx_1 dx_2 ... dx_n$$

Example 1.

For $\mathbf{X}$ and $\mathbf{Y}$ to be independent: $\mathcal{P}_{\mathbf{XY}}(x, y) = \mathcal{P}_{\mathbf{X}}(x)\mathcal{P}_{\mathbf{Y}}(y)$

X and Y are not "independent"

$\mathbf{Y}$ \ $\mathbf{X}$	0	1	$p_{\mathbf{Y}}$
0	0.1	0.2	0.3
1	0.3	0.4	0.7
$p_{\mathbf{X}}$	0.4	0.6	

X and Y are "independent"

$\mathbf{Y}$ \ $\mathbf{X}$	0	1	$p_{\mathbf{Y}}$
0	0.12	0.18	0.3
1	0.28	0.42	0.7
$p_{\mathbf{X}}$	0.4	0.6	

0.1 Condtional CDF, PDF and PMF

Conditional pdf of $\mathbf{Y}$ given $\mathbf{X} = x$ is

$$f_{\mathbf{Y}|\mathbf{X}}(y|x) = \frac{f_{\mathbf{XY}}(x, y)}{f_{\mathbf{X}}(x)} \qquad (3)$$

Conditional pmf of $\mathbf{Y}$ given $\mathbf{X} = x$ is

$$\mathcal{P}_{\mathbf{Y}|\mathbf{X}}(y|x) = \frac{\mathcal{P}_{\mathbf{XY}}(x, y)}{\mathcal{P}_{\mathbf{X}}(x)} \qquad (4)$$

$P_{X|Y}$

Y \ X	0	1
0	1/3	2/3
1	3/7	4/7

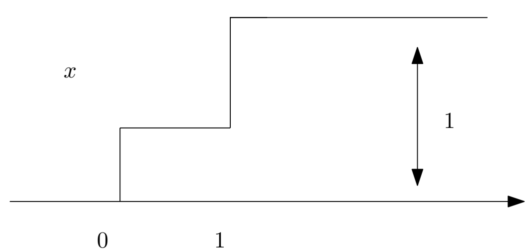
$P_{Y|X}$

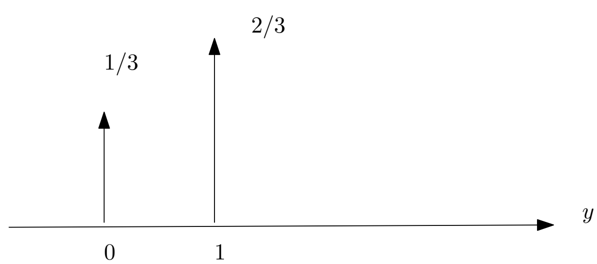
Y \ X	0	1
0	0.12	0.18
1	0.28	0.42

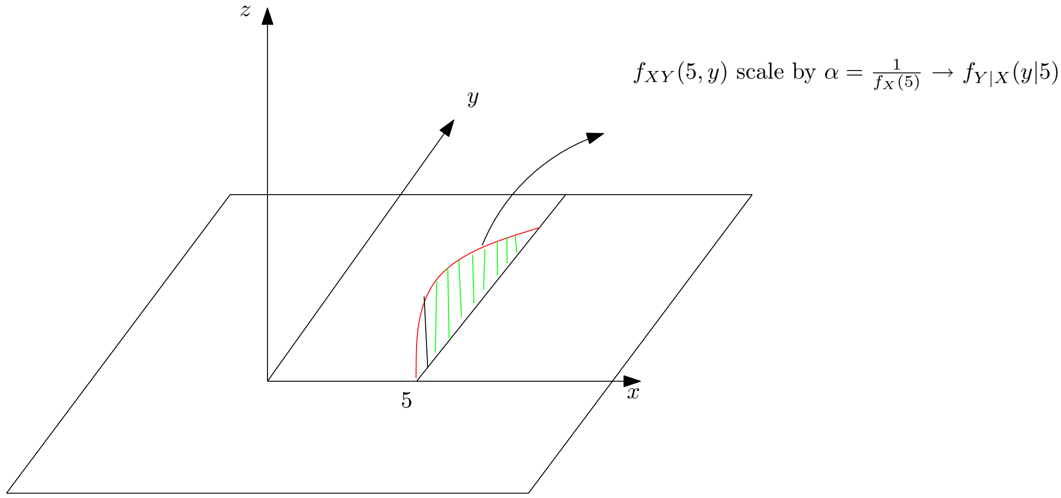
Recall: $\mathcal{P}(A|B) = \frac{\mathcal{P}(A \bigcap B)}{\mathcal{P}(B)}$. From the previouse example

Conditional CDF of **Y** given **X**=x is

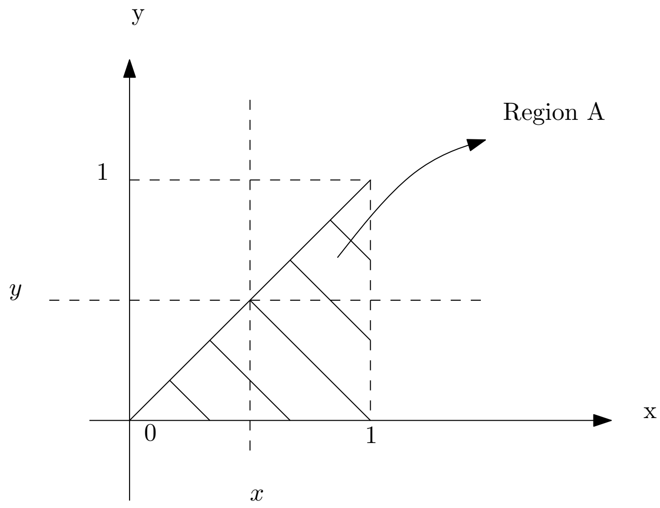
$$F_{\mathbf{Y}|\mathbf{X}} = \int_{-\infty}^{y} f_{\mathbf{Y}|\mathbf{X}}(\alpha|x)d\alpha \qquad (5)$$



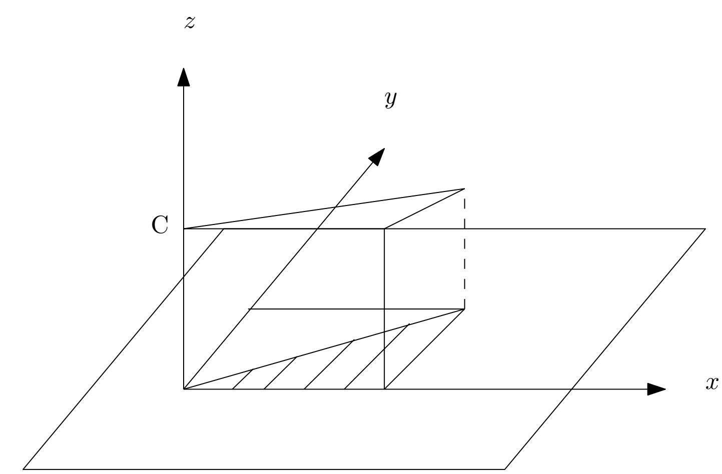



z
y
x
5
$f_{XY}(5, y)$ scale by $\alpha = \frac{1}{f_X(5)} \rightarrow f_{Y|X}(y|5)$

Example 2.



Pick a point from region A at random where each point is drawn equally likely.Find f_{XY}, f_X, $f_{Y|X}$, f_Y.



Clearly

$$f_{\mathbf{XY}}(x,y) = \begin{cases} 0 & , \quad (x,y) \notin A \\ C & , \quad (x,y) \in A \end{cases}$$

for some C.

C needs to be such that: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)dxdy = 1 \Rightarrow C = \frac{1}{1/2} = 2.$

$$f_{\mathbf{XY}}(x,y) = \begin{cases} 0 & , \quad (x,y) \notin A \\ 2 & , \quad (x,y) \in A \end{cases}$$

$$\begin{aligned} f_{\mathbf{X}}(x) &= \int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)dxdy \\ &= \int_{0}^{1} f_{\mathbf{XY}}(x,y)dxdy \\ &= \int_{0}^{x} 2dy \\ &= 2x \quad for \quad x \in [0,1] \end{aligned}$$

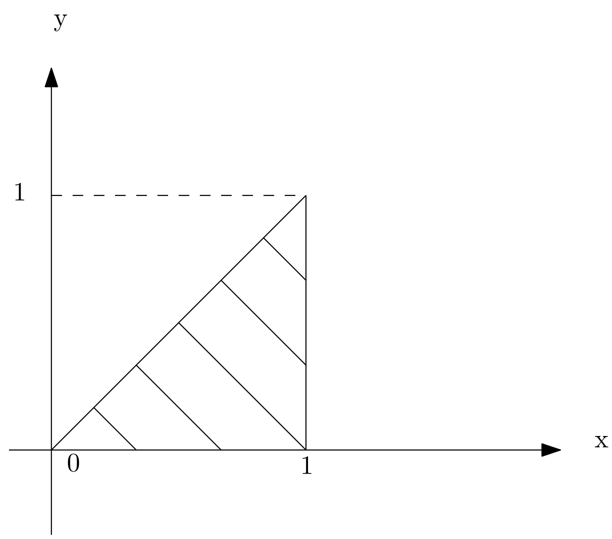
$$f_{\mathbf{X}}(x) = 0 \quad for \quad x \notin [0,1]$$

$$\begin{aligned} f_{\mathbf{X}|\mathbf{Y}}(x|y) &= \frac{f_{\mathbf{XY}}(x,y)}{f_{\mathbf{X}}(x)} \\ &= \begin{cases} \frac{2}{2x} = \frac{1}{x} & , \quad x \in [0,1] \\ 0 & , \quad x \notin [0,1] \end{cases} \end{aligned}$$

$$\begin{aligned} f_{\mathbf{Y}}(y) &= \int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)dx \\ &= \begin{cases} \int_{y}^{1} 2dy & , \quad y \in [0,1] \\ 0 & , \quad y \notin [0,1] \end{cases} \\ &= \begin{cases} 2-2y & , \quad y \in [0,1] \\ 0 & , \quad y \notin [0,1] \end{cases} \end{aligned}$$

Example 3.

Pick a number $\mathbf{X}$ in $[0,1]$ equally likely. Then, pick a number $\mathbf{Y}$ in $[0,\mathbf{X}]$ equally likely. Find $f_{\mathbf{XY}}(x,y)$, $f_{\mathbf{X}}(x)$, $f_{\mathbf{Y}|\mathbf{X}}(y|x)$, and $f_{\mathbf{Y}}(y)$.



$$f_{\mathbf{X}}(x) = \begin{cases} 1 & , \quad x \in [0,1] \\ 0 & , \quad x \notin [0,1] \end{cases}$$

$$f_{\mathbf{Y}|\mathbf{X}}(x) = \begin{cases} \frac{1}{x} & , \quad x \in [0,1] \\ 0 & , \quad x \notin [0,1] \end{cases}$$

$$f_{\mathbf{XY}}(x,y) = f_{\mathbf{X}}(x) f_{\mathbf{Y}|\mathbf{X}}$$

If $x \in [0,1]$

$$\begin{aligned} f_{\mathbf{XY}}(x,y) &= f_{\mathbf{X}}(x) f_{\mathbf{Y}|\mathbf{X}} \\ &= \begin{cases} 1.\frac{1}{x} & , \quad (x,y) \in A \\ 0 & , \quad (x,y) \notin A \end{cases} \end{aligned}$$

If $x \notin [0,1]$ $f_{\mathbf{XY}}(x,y) = 0$

$$\begin{aligned} f_{\mathbf{Y}}(y) &= \int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y) dx \\ &= \begin{cases} \int_y^1 \frac{1}{x} dx & , \quad y \in [0,1] \\ 0 & , \quad y \notin [0,1] \end{cases} \\ &= \begin{cases} -\ln(y) & , \quad y \in [0,1] \\ 0 & , \quad y \notin [0,1] \end{cases} \end{aligned}$$

0.2 Conditional Expectation

$$\mathcal{E}\left[\mathbf{Y}|\mathbf{X}=x\right] := \int_{-\infty}^{\infty} y f_{\mathbf{Y}|\mathbf{X}}(y|x)dy =$$

It follows that $\mathcal{E}\left[\mathbf{Y}\right] = \int_{-\infty}^{\infty} \mathcal{E}\left[\mathbf{Y}|\mathbf{X}=x\right] f_{\mathbf{X}}(x)dx$

$$\begin{aligned}
(\mathcal{E}\left[\mathbf{Y}\right] &= \int_{-\infty}^{\infty} y f_{\mathbf{Y}}(y)dy \\
&= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{\mathbf{XY}}(x,y)dxdy \\
&= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{\mathbf{X}}(x) f_{\mathbf{Y}|\mathbf{X}}(y|x)dxdy \\
&= \int_{-\infty}^{\infty} f_{\mathbf{X}}(x) \left[\int_{-\infty}^{\infty} y f_{\mathbf{Y}|\mathbf{X}}(y|x)dy\right] dx \\
&= \int_{-\infty}^{\infty} \mathcal{E}\left[\mathbf{Y}|\mathbf{X}=x\right] f_{\mathbf{X}}(x)dx)
\end{aligned}$$

In Example 2 $\mathcal{E}\left[\mathbf{Y}|\mathbf{X}=x\right] = \frac{x}{2}$, $\mathcal{E}\left[\mathbf{Y}\right] = \int_0^1 f_{\mathbf{X}}(x)dx = \int_0^1 \frac{x}{2}.2xdx = \frac{1}{3}$

In Example 3 $\mathcal{E}\left[\mathbf{Y}|\mathbf{X}=x\right] = \frac{x}{2}$

$$\begin{aligned}
\mathcal{E}\left[\mathbf{Y}\right] &= \int_0^1 \frac{x}{2}.f_{\mathbf{X}}(x)dx \\
&= \int_0^1 \frac{x}{2}.1dx \\
&= \frac{1}{4}
\end{aligned}$$

If $\mathbf{X}$ and $\mathbf{Y}$ are independent, $\mathcal{E}\left[\mathbf{Y}|\mathbf{X}=x\right]$ does not depend on x, i.e. a constant