Lectures of June 2nd, 2006
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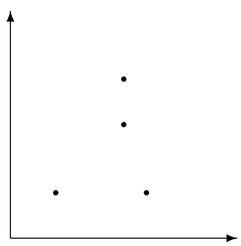
1 Chapter 4

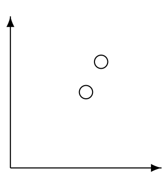
1.1 Multiple Random Variable

A vector random variable (or a random vector) is a function mapping the sample space S of some random experiment into an Euclidean space $\mathbb{R}^m$, where m is the number of random variable in the random vector.

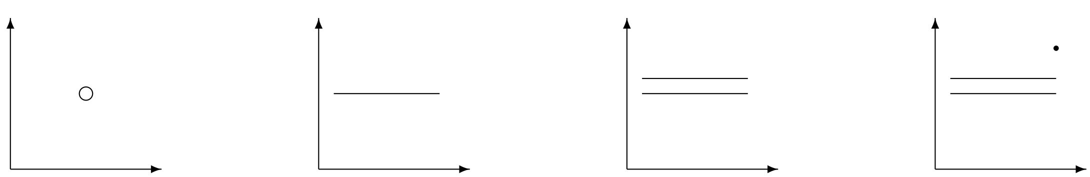
Example: Let (X_1,X_2) be a random vector taking values from the set in the figure, each with equal probability.



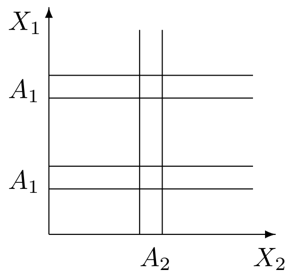
Example: (X_1,X_2) takes value from



Example: (X_1,X_2) takes value from



we often consider event of form $X_1 \in A_1$ and $X_2 \in A_2$ when we consider random vector (X_1,X_2) (A_1 is subset of $\mathbb{R}$, and A_2 is a subset of $\mathbb{R}$)

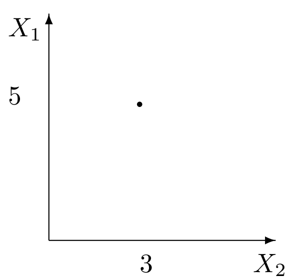


1.2 The (joint) CDF of a random vector

The joint CDF of a random vector $(X_1, X_2, ...X_n)$ is defined by
$F_{X_1X_2...X_n}(x_1, x_2, ..., x_n)$= $P[\mathrm{X}_1 \leq x_1$,$X_2 \leq x_2$,..$\mathrm{X}_n \leq x_n]$

Example: Let RV (X_1,X_2) be such that (X_1,X_2)=(3,5)with probaility 1

Find $F_{X_1X_2}(x_1, x_2)$

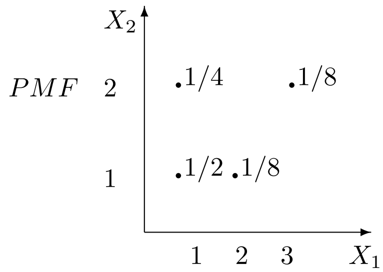


$F_{X_1X_2}(2, 2)$=$P[\mathrm{X}_1 \leq 2$,$X_2 \leq 2$]= 0
$\mathrm{F}_{X_1X_2}(4, 4)$= 0
$F_{X_1X_2}(2, 6)$= 0

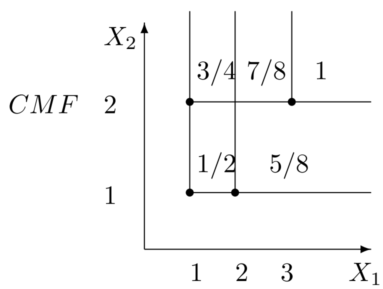
$F_{X_1X_2}(4, 6)$= 1

Solution $\Rightarrow \mathrm{F}_{X_1X_2}(X_1, X_2) = \begin{cases} 1 & X_1 \geq 3,\ \ X_2 \geq 5 \\ 0 & otherwise \end{cases}$

Example2:



Solution:



$$F_{X_1X_2}(x_1,x_2)=\begin{cases} 1 & x_1 \geq 3, x_2 \geq 2 \\ 7/8 & 2 \leq x_1 < 3, x_2 \geq 2 \\ 5/8 & x_1 \geq 2, 1 \leq x_2 < 2 \\ 3/4 & 1 \leq x_1 \leq 2, x_2 \geq 2, \\ 1/2 & 1 \leq x_1 \leq 2, 1 \leq x_2 \leq 0, \\ 0 & \text{otherwise} \end{cases}$$

Properties of joint CDF F_{XY} for any (X,Y)

1) If $x_1 < x_2$, $y_1 < y_2$, then $F_{XY} \leq (x_2, y_2)$
2) For any y, $F_{XY}(-\infty, y) = 0$
For any x, $F_{XY}(x, -\infty) = 0$
3)$F_{XY}(+\infty, +\infty) = 1$
4) $\lim_{x\to a+} \mathrm{F}_{XY}(\mathrm{x,y})=F_{XY}(\mathrm{a,y})$, $\lim_{y\to b+} \mathrm{F}_{XY}(\mathrm{x,y})=F_{XY}(\mathrm{x,b})$

1.3 Definition: Joint PDF

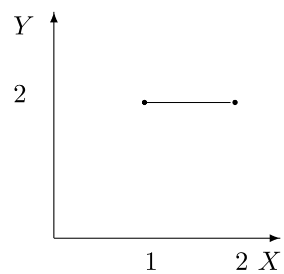
$f x_1, x_2 ... x_n(x_1, x_2 ... x_n)$ of RVs $(X_1, X_2, ... X_n)$ is a non-negative function on R^n such that for any subset A $\leq R^n$.

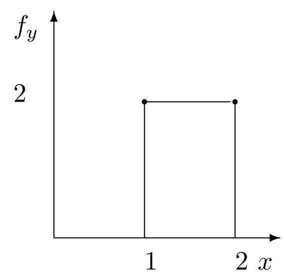
$P[(X_1, X_2, ... X_n) \in A] = \int_A x_1, x_2 ... x_n \, dx$

It can be shown that

$f x_1, x_2 ... x_n(x_1, x_2 ... x_n) = \mathrm{F} x_1, x_2 ... x_n(x_1, x_2 ... x_n) / \ \partial x_1 \partial x_2 ... \partial x_n$

When the right side is well defined $f(x) = \int_a^\infty f(x, y) \, dy$



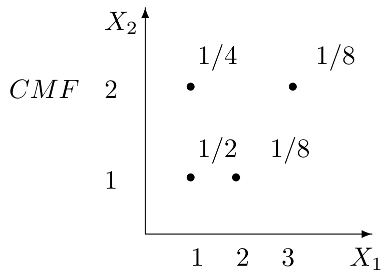


Given (X, Y) $f(x)$ and $f(y)$ are referred to as the marginal PDFs of X and Y.

If every component of $(X_1, X_2, ... X_n)$ is a discrete RV then we may use the joint PMF. $P x_1, x_2 ... x_n(x_1, x_2 ... x_n)$ to represent the distribution of $(X_1, X_2, ... X_n)$.

Specifically: $P x_1, x_2 ... x_n(x_1, x_2 ... x_n) = \mathrm{P}[(X_1, X_2, ... X_n) = (x_1, x_2 ... x_n)]$

Ex: Find $P x_1$ and $P x_2$ for (X_1, X_2) in Example 2.



$$\mathrm{Px}_1(x_1) = \begin{cases} 3/4 & \text{if} \quad x_1 = 1 \\ 1/8 & \text{if} \quad x_1 = 2 \\ 1 & \text{if} \quad x_1 = 3 \\ 0 & \quad\ \ \textit{otherwise} \end{cases}$$

$$\mathrm{Px}_2(x_2) = \begin{cases} 5/8 & \text{if} \quad x_2 = 2 \\ 3/8 & \text{if} \quad x_2 = 2 \\ 0 & \quad\ \ \textit{otherwise} \end{cases}$$

Example: Let the joint PMF $f_{xy}(x, y)$ be

$$\mathrm{f}_{xy}(x, y) = \begin{cases} ab\mathrm{e}^{-ax}e^{-by} & \text{if} \quad x \geq 0 \ y \geq 0 \\ 0 & \quad\ \ \textit{otherwise} \end{cases}$$

where $a \geq 0$ and $b \geq 0$

it can be verified that:

$\int_0^\infty ae^{-ax}\,dx = \int_0^\infty be^{-by}\,dy = 1$

This implies that:

$f_x(x) = ae^{-ax}$ for x ≥ 0

$f_y(y) = ae^{-ay}$ for y ≥ 0

1.4 Independence of random variables

We will denote by A_1, an event involving only X_1, by A_2 an event involving X_2,..., by A_n the event involving only X_n.

If for any A_1, A_2,..., A_n,

P[X_1 ∈A_1 and X_2 ∈A_2 $\ldots X_n$ E A_n]=$\prod_{i=1:n}$ P[X_i ∈A_i]then X_1, X_2,...,X_n are said to be independent:

It can be shown
1)if X and Y are discrete RVs, then if X and Y are independent, then
$P_{XY}(x, y) = P_X(x)P_Y(y)$.
2)if X and Y are continuous, then if X and Y are independent then f_{XY}(x,y)=f_X(x)f_Y(y)
In fact, the converse of the above are also true.
Discrete X and Y are independent ↔P_X(x)P_Y(y)=P_{XY}(x,y)