Lectures of May 26th, 2006
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1 Q Function(continous)

From the last lecture, $Q(x)$ is the right tail of Gaussian PDF with $\mu = 0$ and $\sigma = 1$ starting at x. We also discussed different properties of $Q(x)$ through some examples. If nothing was changed, but the unit of σ and μ, what happens to the right tail area? To have a more clear understanding of the property, the unit of σ was changed from 1mm in figure 1 to 1m in figure 2.

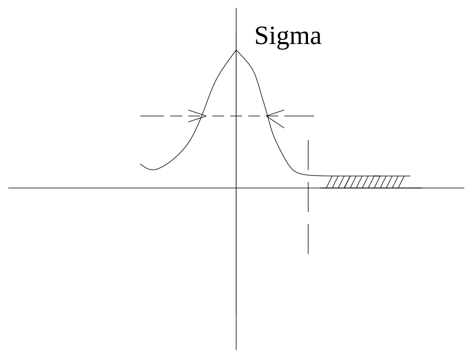

Figure 1: Q function with $\sigma = 1mm$

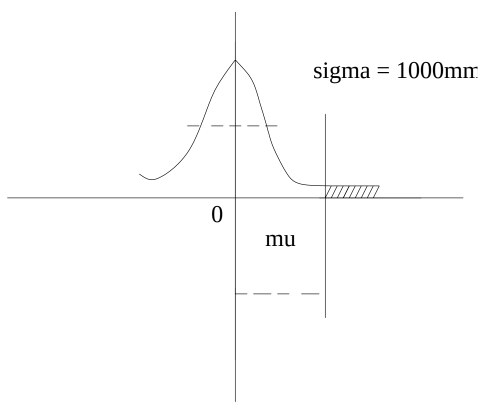

Figure 2: Q function with $\sigma = 1m$

Through observation, the shadow area in figure 1 and 2 won't change regardless the change of the unit of σ. What really matter is the absolute value of σ/μ. If σ/μ change,the area will change.

What if the Q fucntion is right-shifted with d in figure 3

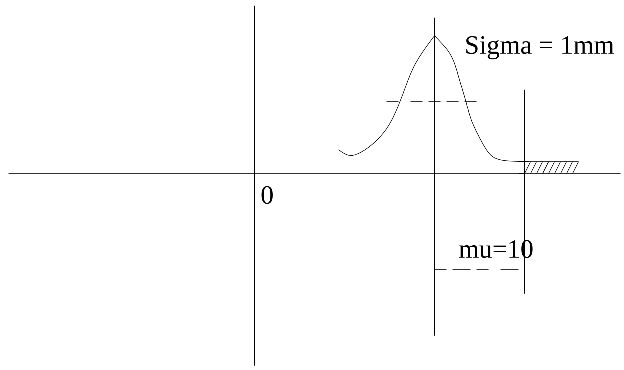

Figure 3: Q function with deviation

Therefore, we can draw a conclusion: for arbitary Gaussian with mean, μ, standard deviation σ, the area of its right tail starting from x is $Q(\frac{x-\mu}{\sigma})$.

2 Functions of a Random Variable

Let X be a random variable, and random variable Y is defined as $Y := g(x)$, for some function g mapping Sx into Sy. That is, every $x \in Sx$ is mapped into some $g(x) \in Sy$.

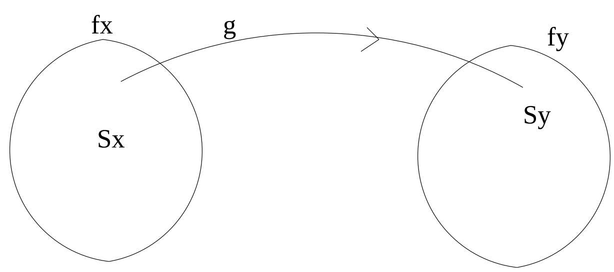

Figure 4: relationship between $S_x and S_y$

For every value X, there is a corresponding y; and the distribution of X induces the distribution of Y.

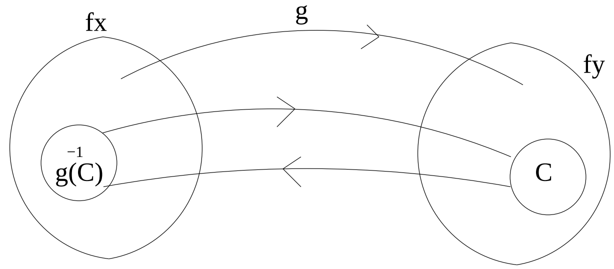

Figure 5: the distribution of $S_y inducedby S_x$

For any sbuset $C \subseteq Sy$, $P[Y \in C] = P[X \in g^{-1}(C)]$

Where $g^{-1}(C)$ is the subset of Sx including all $x \in Sx$ with $g(x) \in C$

Example:

Label the cards in a regular deck by 1,2,3, $\cdots$, 52. Pick a card at random, denote the selected card by X. Then X is a radom variable with

$Sx = \{1, 2, 3, \cdots, 52\}$, and for each $X \in Sx$

$P[X = x] = \frac{1}{52}$

let g be a function define as follows:

$$g(x) = \begin{cases} 0, & \text{if} x > 10, \\ x^2, & \text{otherwise.} \end{cases}$$

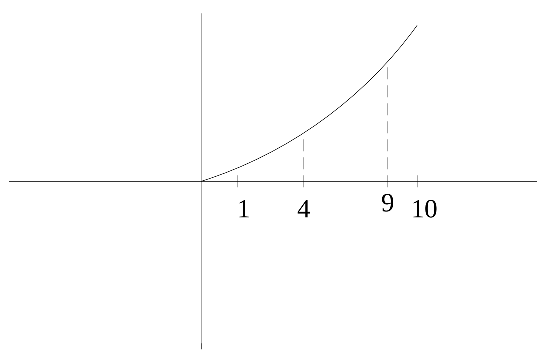

Figure 6: g(x)

let Y be defined as $Y = g(X)$

1) Find $P[Y < 5]$

2) Find $P[Y > 81]$

Solution:

$$\begin{aligned} P[Y < 5] &= P[X > 10] + P[X < \sqrt{5}] \\ &= P[X > 10] + P[\{1, 2\}] \\ &= \frac{42}{52} + \frac{2}{52} \\ &= \frac{11}{13} \end{aligned}$$

$$\begin{aligned} P[Y > 81] &= P[g(x) > 81] \\ &= P[\{X = 10\}] \\ &= \frac{1}{52} \end{aligned}$$

Determining the PDF of a random variable function

Example:

Let X be a discrete random variable with range Sx and PMF $Px(X)$ is defined in the following figure

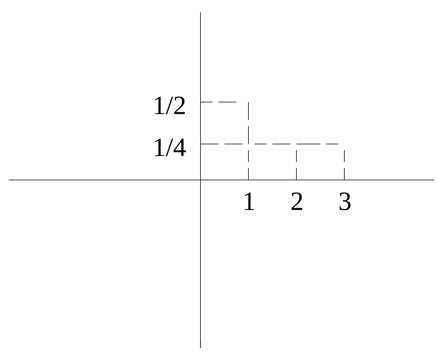

Figure 7: the PMF of Sx

Let random variable Y be defined as $Y = aX + b$ for some given non-zero a and abitary b Find the PMF of Y Solution:

The range of Sy of Y is $Sy = \{a + b, 2a + b, 3a + b\}$ given $a \neq 0$, all the three elements in Sy are distint. Then clearly,

$$\begin{aligned} P[Y = a + b] &= P[X = 1] = 1/2 \\ P[Y = 2a + b] &= P[X = 2] = 1/4 \\ P[Y = 3a + b] &= P[X = 3] = 1/4. \end{aligned}$$

To summize, for any $y = ak + b$, with k = 1,2,3

$Py(y) = Px(k)$

Example:

Let X be a continous random variable with PDF $f_x(x)$, let random variable Y be defined as $Y = aX + b$, for any given non-zero a and some arbitary b. Express the PDF $f_y(y)$ in terms of the PDF $f_x(x)$ of X.

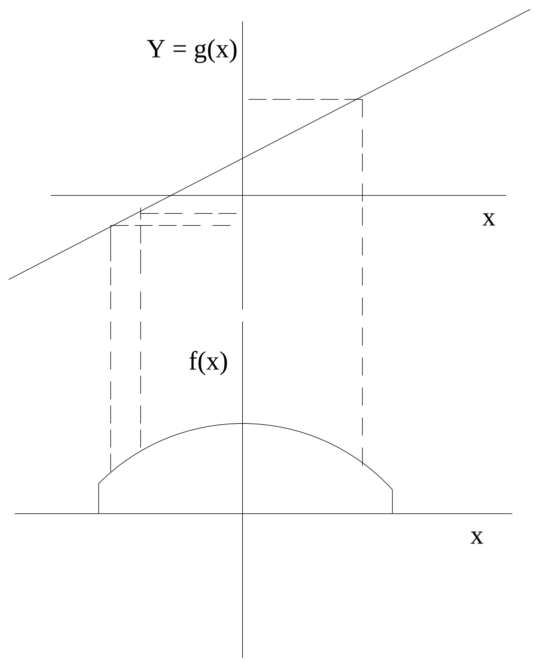

Figure 8: The PDF of X and the linear function $y = aX + b$

Solution:

Since $Y = aX + b$ is a linear function, the shape of PDF of $f_y(y)$ should be the same as the $f_x(x)$ with some scale change and shifting

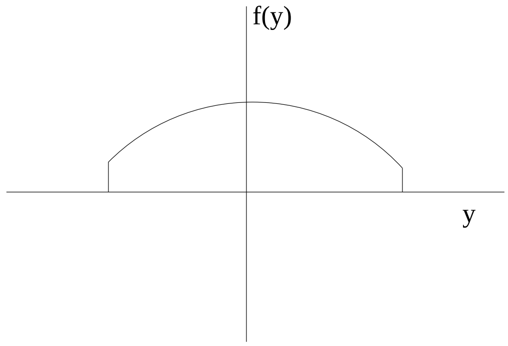

Figure 9: the PDF of S_y

We will find first the CDF $F_Y(y)$ of Y

$$\begin{aligned} F_y(y) &= P[Y \leq y] \\ &= P[ax + b \leq y] \end{aligned}$$

$$F_y(y) = \begin{cases} P[X \leq \frac{y-b}{a}], & a > 0, \\ P[X \geq \frac{y-b}{a}], & \text{a} < 0. \end{cases}$$

If $a > 0$

$F_y(y) = F_x(\frac{y-b}{a})$

then, the PDF $f_y(y)$ is

$$\begin{aligned} f_y(y) &= F'_y(y) \\ &= F'_x(\frac{y-b}{a}) \\ &= \frac{1}{a} f_x(\frac{y-b}{a}) \end{aligned}$$

If $a < 0$,

$$\begin{aligned} F_Y(y) &= 1 - P[X < (\frac{y-b}{a})] \\ &= 1 - P[X \leq (\frac{y-b}{a})] + P[X = \frac{y-b}{a}] \\ &= 1 - F_x(\frac{y-b}{a}) + 0 \\ &= 1 - F_x(\frac{y-b}{a}) \end{aligned}$$

Then the PDF $f_y(y)$ of Y is

$$\begin{aligned} F_y'(y) &= 1 - F_x(\frac{y-b}{a}) \\ &= -\frac{1}{a} f_x(\frac{y-b}{a}) \end{aligned}$$

To summarize, in general

$$F_y(y) = |\frac{1}{a}| f_x(\frac{y-b}{a})$$

Example:

Let the PDF f_x of X be as follows

$$f_x(x) = \begin{cases} 1/10, & -5 < \mathrm{x} < 5, \\ 0, & \text{otherwise.} \end{cases}$$

Let $g(x)$ be another function

$$g_x(x) = \begin{cases} x+1, & -\infty < \mathrm{x} < \infty, \\ 0, & -1 \leq \mathrm{x} \leq 1, \\ x-1, & \mathrm{x} > 1 \end{cases}$$

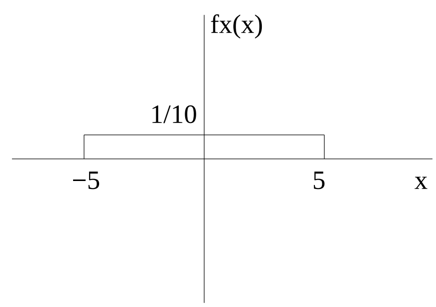

Figure 10: the PDF of f_x of X

Let $Y = g(x)$, Find PDF f_y of Y

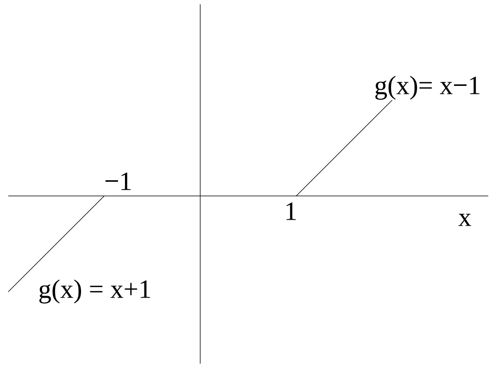

Figure 11: $g(x)$

Solution:

When $y < -1$

$$F_y(y) = |\frac{1}{a}|f_x(\frac{y-b}{a})$$

$a = 1$ and $b = 1$

$f_y(y) = f_x(y-1)$

$-5 \leq x \leq -1$

$-5 \leq y-1 \leq -1$

Therefore $-4 \leq y \leq 0$

$f_y(y) = f_x(x) = \frac{1}{10}$

When $-1 \leq x \leq 1$

$y = 0$

$f_y(y) = 0.2\delta(t)$

When $y > 1$

$$F_y(y) = |\frac{1}{a}|f_x(\frac{y-b}{a})$$

$a = 1$ and $b = -1$

$f_y(y) = f_x(y+1)$

$1 \leq x \leq 5$ $1 \leq y+1 \leq 5$

$0 \leq y \leq 4$

$f_y(y) = f_x(x) = \frac{1}{10}$

To summarize,

$$f_y(y) = \begin{cases} 1/10 + 0.2\delta(t), & -4 \le \text{y} \le 4, \\ 0, & \text{otherwise} \end{cases}$$

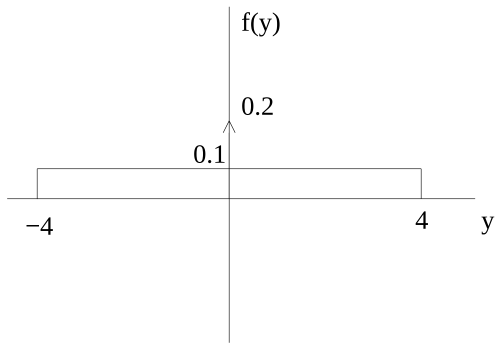

Figure 12: $f_y(y)$