Lectures of May 25th, 2004
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1 Understanding α and scaling

Example 1 *If the probability of getting calls is 1.3 on average over 2 hours. Whats the probability of getting 3 calls over a 5 hour interval?*

Given that the following assumptions are true:
- Independence along time
- Equal likelihood on each time-point

Solution

Over 2 hour interval:
Probability is: $P_X(k=3)$ *with* $\alpha = 1.3$

Over 5 hour interval:
Probability is: $P_X(k=3)$ *but with a new adjusted* α *, so that:*
$\alpha\ = (1.3).(\frac{2}{5})$

2 Continuing the 'Big Example' from previous lecture

2.5-a

With the same assumption that every interval of length 2 has α phone calls on average, consider the continuos version of 2.3. i.e. When will the first phone call arrive?

Solution

Let the answer be X, wich is a RV. Note: X is a RV.

In 2.3, for any positive integer k, we have:

$$P[\text{X} > \text{k}_o] = P[\text{no call hits the first k intergal}] = (1 - \text{P})^k$$

When Δ is small enough, any time t can be appointed by

$$\text{t} = \text{k}\Delta, \text{ for some k}$$

Then,

$$P[X > 1] = (1 - P)^{(\frac{t}{\Delta})} = (1 - (\frac{\alpha}{n}))^{(\frac{t}{\Delta})} =(1 - (\frac{\alpha}{n}))^{(\frac{tn}{2})}$$

where, n = number of intervals of Δ in 2 hours distribution

As n$\rightarrow \infty$, $(\exp^{-\alpha})^{(\frac{t}{2})} = (\exp)^{(-\frac{\alpha}{2}t)}$

note: $\frac{\alpha}{2} := \lambda$, average number of calls arriving over any duration of 1 time unit (in this case 1 hour)

The CDF of X is then,

$$P_X(x) = P[x \leq \text{k}] = 1 - \text{P[X > x]} = 1 - \exp^{-\lambda x}, for\ \text{x} \geq 0$$

The pdf of X is then,

$$f_X(x) = \lambda \exp^{-\lambda x}, \text{ for x} \geq 0$$
$$= 0, otherwise$$

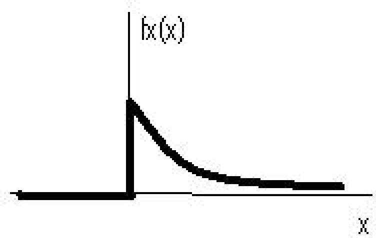

Figure 1. The PDF of X

This example shows that, with the 2 mentioned assumptions, we will have to end up with and exponential distribution of the RV in terms of time.

Remarks:

	Discrete-time (in terms of # of trials)	Continuous-time (in terms of # of trials)
Order of n trials [how many]	***Binomial RV***	***Poission RV***
How long to wait until the first successful trial (ex."Heads"/"call") [When]	***Geometric RV***	***Exponential RV***

2.5-b

Suppose that 5 hours have passed, and there were no calls, when will the fist call arrive?

Solution:

The answer X is a RV having te following "memoryless" property:

$$P[x > 5 + b|x > 5] = P[x > b], foranypositveb$$

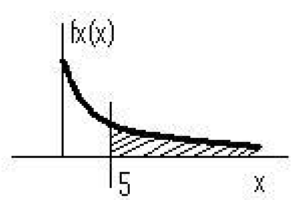

Figure 2. The PDF of X

The PDF is:

$$f_{x|x>5}(x) = \beta f_x(x-5) forsome \beta :_{x|x>5} (x)dx = 1$$

Note in this example the fact that no calls arrive untill x=5 does not mean we will get one very soon (Independence). So if we scale the conditional PDF above so that the total area = 1, we will get the original PDF.

3 Continuing Types of Random Variables

6. Gaussian RV

$\rightarrow ContinousRV$
$\rightarrow S_x = (-\infty, +\infty)$
$\rightarrow$A Gaussian RV is specified by the following PDF:

$$\mathrm{f}_x(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Parametrized by:

-μ : The mean (average) of the RV
-σ : The standard deviation of the RV

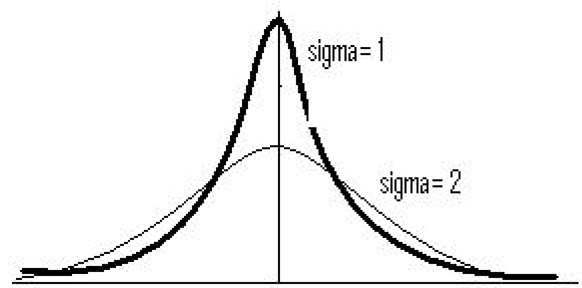

Figure 2. PDF of a Gaussian RV

The Gaussian RV is typically used to model the sum of many independent RV's or simply assumed for conveniance.

Q Functions

For any x:

$$Q(x) = \int \frac{1}{\sqrt{2\pi}} \exp^{\frac{-x^2}{2}}$$

That is, Q(x) is the rite tail of the Gaussian PDF with $\sigma = 1$

Example 2 *Suppose we chose the meter(m) to be the unit of distance. Suppose the position of a bug on the x-axiz is a Gaussing RV with* $\mu = 0 and \sigma 1m^2$

i.e.

$$f_x(x) = \frac{1}{\sqrt{2\pi}} \exp^{\frac{-x^2}{2}}$$

Find the probability that the bug is more than 2 meters to the rite of the origin

.

Answer is Q(2)

Now suppose that we change the unit of distance to mm. Answer the same question aain.

Clearly the answer is still Q(2) but the distribution is different.

Noe: the PDF expressed using the 1 mm is different:

$$f_x(x) = \frac{1}{\sqrt{2\pi}100} \exp^{\frac{-x^2}{2.1000}} dx$$

4 Mean and Variance of a RV

Suppose $f_x(x)$ is the PDF of a RV X, then:

- The mean (also called the expected value) of x is defined as:

$$E[x] = \int_{-\infty}^{+\infty} x f_X(x) dx$$

- The variance of X is defined as:

$$VAR[X] = E[X] = \int_{-\infty}^{+\infty} (x - E[x])^2 f_X(x) dx$$

- The standard deviation of X is defined as:

$$STD[X] = \sqrt{VAR[X]}$$

Example 3

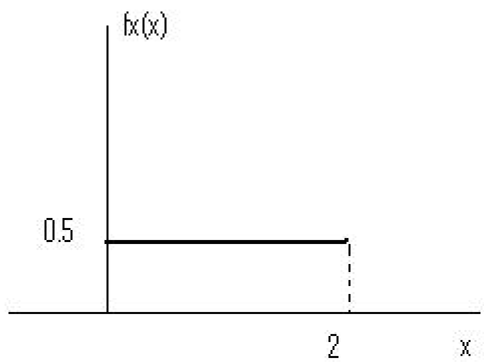

Figure 4. *The PDF of X*

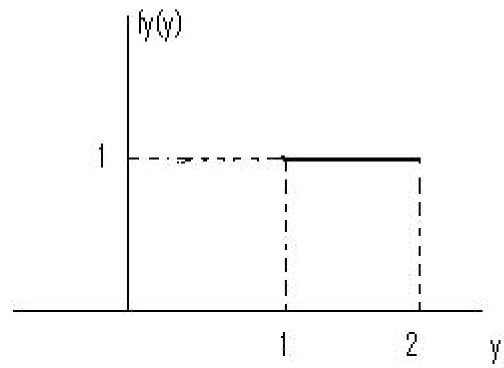

Figure 5. *The PDF of X*

Let X and Y be two RV's with PDF given it the figures above. Now we will generate RV Z as follows"'

Toss a coin with P[H] = 0.3

if the outcome is H, draw a number X and let $z = x$ if th eoutcome is T, draw a number Y and let $z = y$

Find the mean of Z.

$$P[Z = z] = P[Z = z|C = H].P[C = H] + P[Z = z|C = T].PC = T$$
$$F_Z(z) = f_x(z)$$

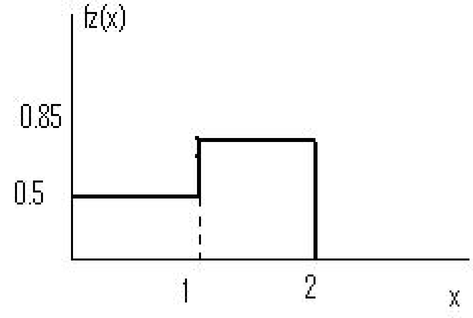

Figure 5. *The combined PDF of X and Y with total area = 1*

$$E[x] = \int_{-\infty}^{+\infty} f_Z(z)dz = \int_0^1 0.15z dz + \int_1^2 0.85z dz$$
$$STD[X] = \int_{-\infty}^{+\infty} [X - E[x]^2 f_x dx$$