Lectures of ***May 19, 2006 - 8:00 AM***
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1 Cummulative Distribution Function (CDF) Of A Random Variable (RV)

CDF F_X of RV X is defined as:

$$F_X(x) = P\left[X \leq x\right], \qquad \forall x \in \mathbf{R}$$

CDF F_X(x) is a function mapping $\mathbf{R}$ to the interval $[0, 1]$.

Example 1: *Toss a fair coin 3 times independently, and let a Random Variable X be the number of "H" seen. Determine the CDF of X.*

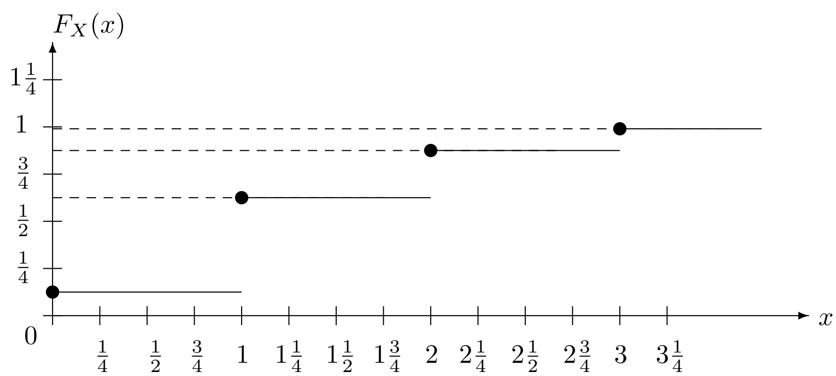


Figure 1

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & \frac{1}{8} \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{7}{8}, & 2 \leq x < 3 \\ 1, & x > 3 \end{cases}$$

1.1 Properties of CDF

1. $0 \leq F_X(x) \leq 1, \qquad \forall x \in \mathbf{R}$
2. $\lim_{x \to +\infty} F_X(x) = 1$
3. $\lim_{x \to -\infty} F_X(x) = 0$
4. $F_X(x)$ is a non-decreasing function (i.e. $\forall \quad x_1, x_2 \in \mathbf{R}$). However, if $x_1 < x_2$, then $F_X(x) \leq f_X(x_2)$.
5. $F_x(x)$ is continuous from the right, (i.e. $\forall \quad b \in \mathbf{R}$,
   $F_X(b) = \lim_{x \to 0^+} F_X(b + h)$).

Example 2: *Pick a point from a unit circle on x-y plane, equally likely. Let RV X be defined as, Z = ∠ (x,y) for any picked (x,y). The range of Z is considered to be [0,2π).*

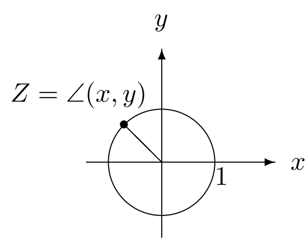


Figure 2

Find: *CDF $F_Z(z)$.*

Solution:
$F_Z(z) = P[Z \leq z] = 0$ if $z < 0$. However, if we consider z = 2 rads. Then, $F_Z(z) = P[Z \leq 2rad] = \frac{2rad}{2\pi}$. That is, for any z $\in [0, 2\pi)$, $F_Z(z) = \frac{z}{2\pi}$. In other-words,

$$F_Z(z) = \begin{cases} 0, & z < 0 \\ \frac{z}{2\pi}, & 0 \leq z \leq 2\pi \\ 1, & otherwise \end{cases}$$

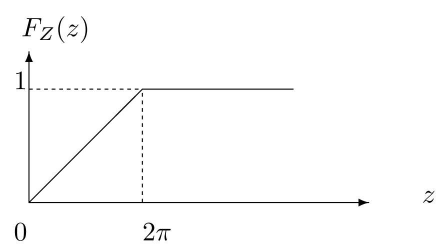


Figure 3

2 Probability Density Function (PDF) Of A RV

Given a RV X, the PDF $f_X(x)$ of X is defind as, $f_X(x) = \frac{dF_X(x)}{dx}$.

Example 3: *Consider example 2 where* $F_Z(z)$ *is*

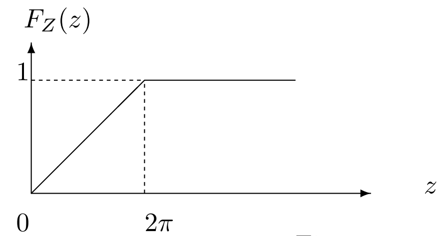


Figure 3

Find: $f_Z(z)$

Solution:

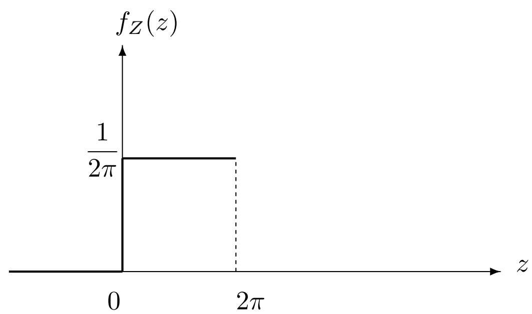


Figure 4

Remarks: *The integration of the PDF of* $f_Z(z)$ *indicates the likelihood of event* $Z = z$. *In this example,* $f_Z(z)$ *is constant for* $0 < z < 2\pi$, *and hence it is equally likely to occur (by definition of PDF).*

Exercise: *Consider the following equation:*

$$\begin{aligned}
\int_{a^+}^{b} f_X(x)dx &= \int_{-\infty}^{b} f_X(x)dx - \int_{-\infty}^{a} f_X(x)dx \\
&= F_X(b) - F_X(a) \\
&= P[X \leq b] - P[X \leq a] \\
&= P[a \leq X \leq b]
\end{aligned}$$

$\vdots$

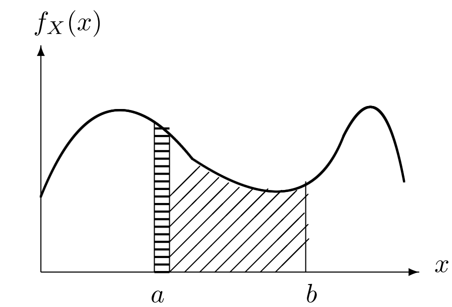


Figure 5

We note that: $\int_{-\infty}^{+\infty} f_X(x)dx = 1$

Example 4*: Suppose that the CDF, $F_X(x)$, of RV X is given by the figure below,*

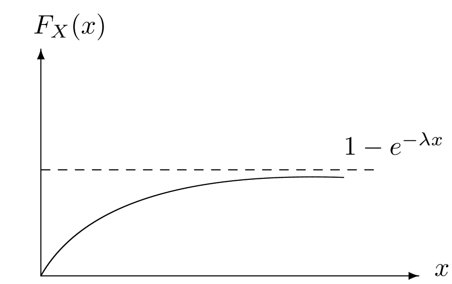


Figure 6

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases} \quad \text{for any given } \lambda$$

Find: $f_X(x)$.

Solution:
If $x < 0$, $f_X(x) = 0$, on the other hand, if x ≥ 0, $f_X(x) = \frac{d(1-e^{-\lambda x})}{dx} = \lambda e^{-\lambda x}$

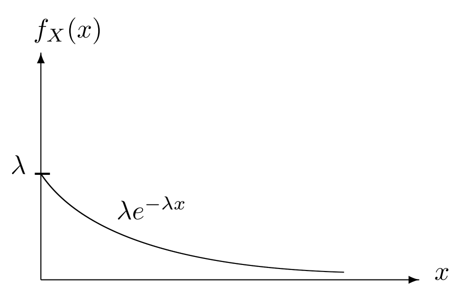


Figure 7

We need to compute PDF from discrete CDFs. Recall from ***Signal and System Theory***, $\delta(x)$ is defined as follows:

1. if x≠0, δ(x)= 0,
2. $\int_{-\infty}^{+\infty} \delta(x)dx = 1$

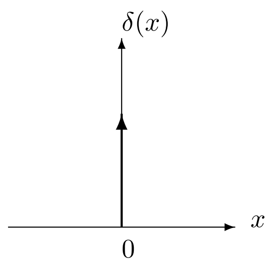


Figure 8

$\int_{-\infty}^{t} \delta(x)dx := \mu(t)$

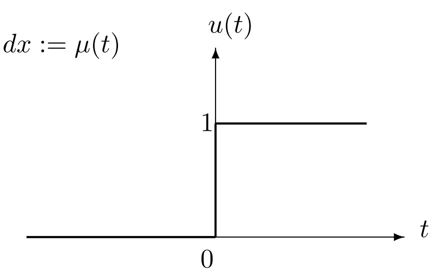


Figure 9

$\frac{du(t-5)}{dt} = \delta$(t-5)

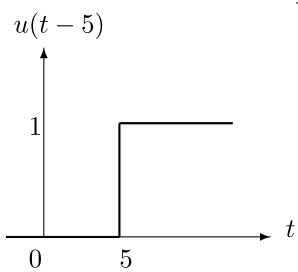


Figure 10

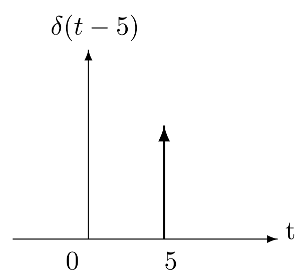
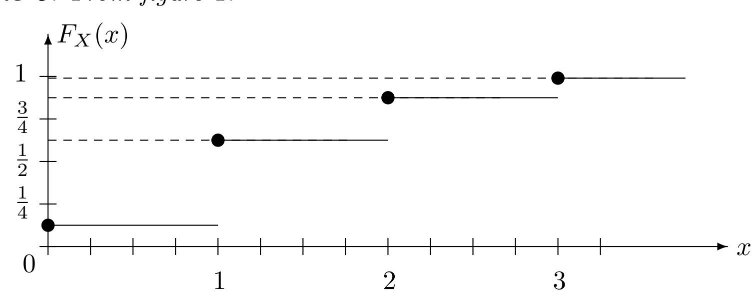


Figure 11

Example 5: *From figure 1.*



This CDF can be expressed as:

$F_X(x) = \frac{1}{8}\mu(x) + \frac{3}{8}\mu(x-1) + \frac{3}{8}\mu(x-2) + \frac{1}{8}(x-3)$. *However, we can find the PDF version of $F_X(x)$ by simply using the definition of PDF. Graphically, $F_X(x)$ is the summation of the following step functions.*

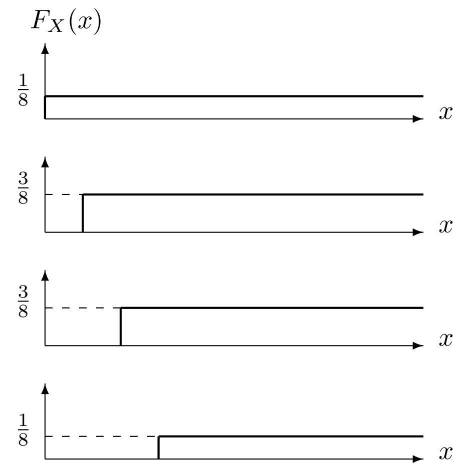


Figure 12

$$
\begin{aligned}
f_X(x) &= \frac{dF_X(x)}{dx} \\
&= \frac{1}{8}\delta(x) + \frac{3}{8}\delta(x-1) + \frac{3}{8}\delta(x-2) + \frac{1}{8}\delta(x-3)
\end{aligned}
$$

However, if we graph $f_X(x)$, it would look like:

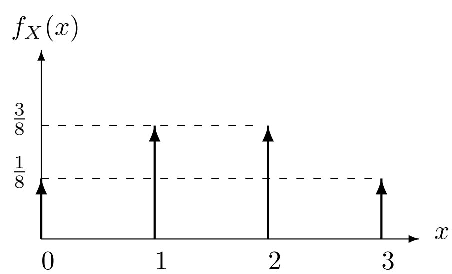


Figure 13

Observations:

- *The integration of this function over (-∞,+∞).*
- *The coefficient for any $\delta(x - \alpha)$ is precisely the probability of $X = \alpha$.*

Example 6: *Given $f_X(x) = T(x) + \alpha\delta(x-2)$ where $f_X(x)$ and $T(x)$ are shown below in figures 14 and 15.*

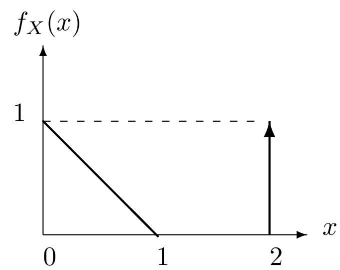


Figure 14

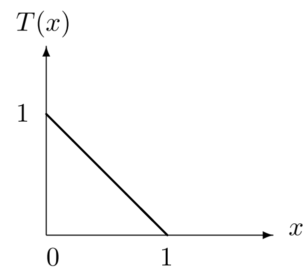


Figure 15

Find :

1) α

2) CDF of $f_X(x)$.

Solution:

1) Since $\int_{-\infty}^{+\infty} f_X(x)dx = 1 = \frac{1}{2} + \alpha \int_{1.9}^{2.1} \delta(x-2)dx \rightarrow \alpha = \frac{1}{2}$.

2)

$$\begin{aligned} F_X(t) &= \int_{-\infty}^{t} f_X(x)dx \\ &= \int_{-\infty}^{t} T(x)dx + \int_{-\infty}^{t} \frac{1}{2}\delta(x-2)dx \end{aligned}$$

$$T(x) = \begin{cases} 1-x, & x \in [0,1] \\ 0, & otherwise \end{cases}$$

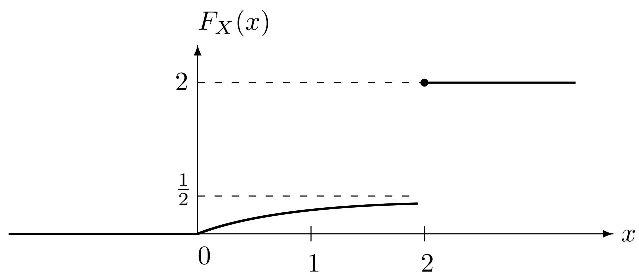


Figure 16

Remarks:

- *If the PDF of a RV is continuous then the RV is said to be continuous.*
- *If the PDF of a RV consists of only Delta (δ) functions, the RV is said to be discrete.*
- *If a RV is neither discrete nor continuous, it is said to be mixed-type.*

In case of discrete RVs, it is clear that PDF $f_X(x)$ can be expressed as:

$$f_X(x) = \sum_{k=1}^{+\infty} P(x_k)\delta(x - x_k), \qquad \textit{then}$$

$\{ P(x_1), P(x_2), \cdots, P(x_k)\}$ *is referred to as the* ***probability mass function (PMF) of X***.

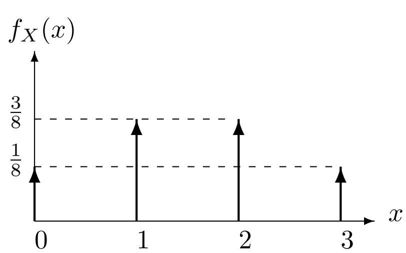


Figure 17

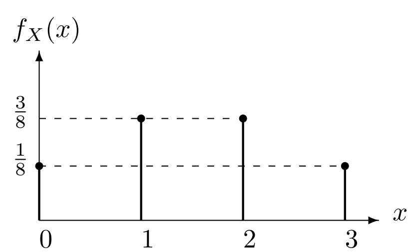


Figure 18

NB: *Figure 17 represents a probability density function while Figure 18 represents a probability mass function which only exists in discrete representation!*