Lectures of May 16th, 2006
Scribe: Mirza Abdel Jabar Baig, Raza Hanif, Michael Mackie

Geometric Probability Law[1]

SETUP - Keep doing some independent Bernoulli trials (with probability of success equal to P), until the first success. Note the number of trials.

$S = \{1, 2, ...\}$

$P[\{1\}] = p$

$P[\{2\}] = (1-p)p$

$P[\{3\}] = (1-p)^2 p$

$P[\{k\}] = (1-p)^{k-1}p, \forall k \in S$

Verify $\sum_{k=1}^{\infty} P[\{k\}] = 1$ (sum adds to 1 by geometric series convergence)

To find the probability we need k trials: $1 - P[\{k\}]$

Example 1 *Suppose that a batch of CPUs have a defect rate of* 10^{-5}.

[We interpret this as that the probability that randomly chosen CPU from the batch is defective is 10^{-5}*.]*

We are now checking these CPUs, one by one, one at a time.

1. *What is the probability that the* 100^{th} *CPU has a defect?*
2. *What is the probability that the first found defective CPU is the* 100^{th}*?*
3. *What is the probability that there is precisely one defective CPU in the first 100 checked?*
4. *What is the probability that there is at least one defective CPU in the first 100?*
5. *Suppose that we have checked 99 CPUs and they are all good, what is the probability that the next one has a defect?*

Solution

1. 10^{-5}, since we refer to a particular Burnoulli trial which by defintion has $P = 10^{-5}$.
2. $(1 - 10^{-5})^{99} 10^{-5}$, by the geometric law.

[1] Chapter 2, page 66

3. $\binom{100}{1}(1-10^{-5})^{99}10^{-5} = 100(1-10^{-5})^{99}10^{-5}$, by the binomial law.

4. ***Solution 1-*** The probability that there are preceisely k defect CPUs is

$$\binom{100}{k}(1-10^{-5})^{100-k}(10^{-5})^k$$

The probability of interest in the question is

$$\binom{100}{1}(1-10^{-5})^{99}(10^{-5})^1+\binom{100}{2}(1-10^{-5})^{98}(10^{-5})^2+...+\binom{100}{100}(1-10^{-5})^0(10^{-5})^{100}$$

$= \sum_{k=1}^{100}\binom{100}{k}(1-10^{-5})^{100-k}(10^{-5})^k$

Solution 2- The probability of having no defect is $(1-10^{-5})^{100-k}$. Probability of at least one defect is $1-(1-10^{-5})^{100-k}$, which is the compliment event.

5. 10^{-5}, due to the assumptions of independent experiments.

Example 2 *Use a fair coin to create a random number. Take value 9 with probability 0.25, and 12 with probability 0.75.*

Solution

Toss the coin twice. If HH then the output is 9. Otherwise, the output is 12.

Example 3 *Use a fair coin to create a random number taking values from* $\{0, 2^{-8}, 2x2^{-8}, \ldots, (2^8-1)2^{-8}\}$*, each with equal probability.*

Solution

Toss the coin 8 times. For each output, output a distinct number in

$\{0, 2^{-8}, 2x2^{-8}, \ldots, (2^8-1)2^{-8}\}$.

Specifically, the following way of assigning real number to a win-tossing output as follows:

Denote the output of the i^{th} coin-tossing by X_n, where we will use “0” to represent H, and “1” to represent T. The real number can be generated as

$$X_1(2^{-1}) + X_2(2^{-2}) + X_3(2^{-3}) + ... + X_8(2^{-8})$$

It is easy to verify

1. the largest possible number generated is $(2^8-1)2^{-8}$.
2. the smallest possible number generated is 0.
3. every generated number is an integer multiple of 2^{-8}.

This verfies that random number we generate satisfies the question.

Random Variables[2]

DEFINTION - A random variable X is a function that maps the sample space S of a random experiment into the set $\mathbb{R}$ of real number. The sample space S is called the domain of a RV, and the set of all values that X may take is called the range of RV denoted by S_x. Clearly $S_x \subseteq \mathbb{R}$.

Example 1 *Consider coin-tossing experiment where*

$$S = \{H, T\}$$

Define a RV as follows: for any $\zeta \in S$

$$X(\zeta) = \left\{ \begin{array}{ll} 1, & if \ \ \varsigma = H \\ 0, & if \ \ \varsigma = T \end{array} \right\}$$

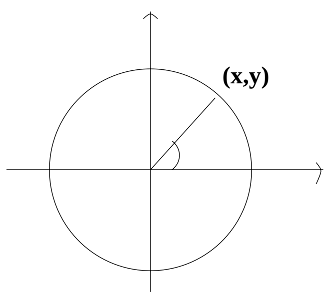
Example 2 *Toss a coin three times, $S = \{HHH, ..., TTT\}$.*

Define a RV X as follows: for any $\zeta \in$ S,

$$X(\zeta) := \text{the number of "H" in } \zeta$$

The range of X is $S_x = \{0, 1, 2, 3\}$

Example 3 *Define a random experiment by picking a point on the unit circle in the x-y plane.*



$$S = \{(x, y) : x^2 + y^2 = 1\}$$

Define a RV Z as follows for every (x,y)$\in$ S

$$Z(x, y) = \angle(x, y)$$

The range of Z is $S_z = [0, 2\pi)$

- When the sample space S of a random experiment is a subset of $\mathbb{R}$, we may take the outcome of the random experiment as a random variable X, where $X(\zeta) = \zeta$, for any $\zeta \in$ S.

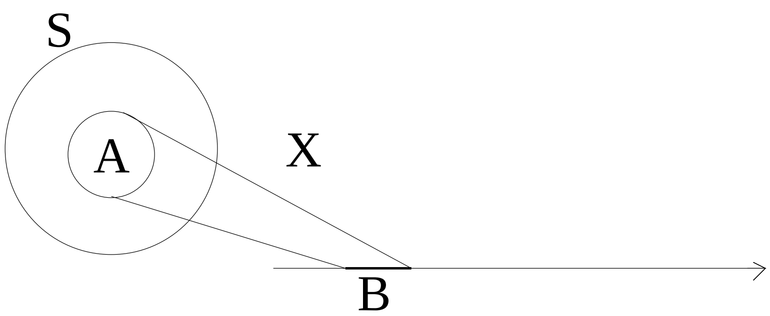
[2]Chapter 3

- We often write $X(\zeta)$ as X for simplicity, but it is useful to keep in mind, X is a function.

Example 4 *Toss a coin 3 times, $S = \{HHH, \ldots, TTT\}$(equally likely). Let $X(\zeta)$ be the number of "H" in ζ, $\forall \zeta \in S$. What is the probability $X \leq 2$?*

Solution

$$F_x = P[X \leq 2] = P[\{TTT, TTH, HTT, THH, HTH, HHT\}] = \frac{7}{8}$$



In the general setting $P[X \in B] = P[\zeta \in A]$, when A is the set of all $\zeta \in S$ such that $X[\zeta] \in B$. Given $B \subseteq S_x$, $A := X^{-1}(B)$ is sometimes referred to as the equivalent in S to B.

Cumalative Distribution Function (CDF) of RV

Given a random variable X, the CDF of X, denoted by $F_x(X)$ is a function defined as

$$F_x(X) = P[X \subseteq x], \forall x \in \mathbb{R}$$

Example 5 *Toss a coin 3 times, $S = \{HHH, \ldots, TTT\}$(equally likely). Let $X(\zeta)$ be the number of "H" in ζ, $\forall \zeta \in S$. What is the probability $X \leq 2$?*

Solution

$$F_x = P[X \subseteq x]$$