Lecture of May 12th 2006, PM Class
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Independance of multiple events[1]

(follow up from Lecture 4)

Recall:

1. Idependence: $P[A \cap B] = P[A] \cdot P[B]$
2. Dependence: $P[A \cap B] = P[A] \cdot P[B|A]$

Note: It is important to notice that independency is a special case of dependency.

More about independent experiments

Let's consider the following:

Exp1 : S_1 with a probability law P[ ] specified.

Exp2 : S_2 with a probability law Q[ ] specified.

Now if Exp1 and Exp2 are independent, then a "combined" experiment C may be defined by first performing Exp1 and then performing Exp2. The sample space of the combined experiment C is:

$$S_C = S_1 \ X \ S_2$$

Then, the probability law R[ ] of Exp C, assuming that A_1 and A_2 are events of S_C, is the following:

$$R[A_1 \ and \ A_2] = R[A_1 \ X \ A_2] = P[A_1] \cdot Q[A_2], \text{ for any } A_1 \subseteq S_1, \ A_2 \subseteq S_2$$

Note: $R[A_1 \cap A_2]$ is not well defined because both events have no elements in common. Instead, we must consider the following:

$$A_1 \ X \ A_2 = (A_1 \ \times \ S_2) \cap (S_1 \ X \ A_2)$$

[1] 2.5 Independence of events, P. 54 of the textbook

Example 1: Proof

Setup:

$$\begin{aligned} S_1 &= \{1,2,3\} \\ A_1 &= \{1,2\} \subseteq S_1 \\ S_2 &= \{J,Q,K\} \\ A_2 &= \{J,Q\} \subseteq S_2 \end{aligned}$$

Using the cartesian product, we find the following:

$$S_1\ X\ S_2 = \{1J, 1Q, 2J, 2Q, 3J, 3Q\}$$

Also, in the same fashion, we find:

$$\begin{aligned} A_1 \times A_2 &= \{1J, 1Q, 2J, 2Q\} \\ A_1 \times S_2 &= \{1J, 1Q, 1K, 2J, 2Q, 2K\} \\ S_1 \times A_2 &= \{1J, 1Q, 1K, 2J, 2Q, 2K, 3J, 3Q, 3K\} \end{aligned}$$

Therefore, recalling that $A_1\ X\ A_2 = (A_1\ X\ S_2) \cap (S_1\ X\ A_2)$, we have the following:

$$(A_1\ X\ S_2) \cap (S_1\ X\ A_2) = \{1J, 1Q, 2J, 2Q\}$$

Example 2:

A prisoner is given a last chance to decide whether he will be executed. He is given two identical jars, 100 red balls and 100 black balls. He can put the balls into the two jars in any way he wants but every ball has to be in a jar. Then, he is blindfolded. The jars are shaked and possibly have their positions changed. Then, the prisoner is asked to pick a ball. If he picks a black ball, he will be executed. Otherwise, if he picks a red ball, he will be released. How should he put the balls into the jars so as to maximize his chances of living ?

Note: Clearly, the prisoner could put all red balls into jar 1 and all black balls into jar 2 which would bring his chances of living up to 50%. But can he arrange the balls in a way that he can have a greater probability of living ?

Solution: The prisoner should put a single red ball into jar 1 and all other balls into the jar 2. Here is the demonstration:

$$\begin{aligned}
P[Jar1] &= P[Jar2] = 1/2 \\
P[Red \mid Jar1] &= 1 \\
P[Black \mid Jar1] &= 0 \\
P[Red \mid Jar2] &= 99 \cdot \frac{1}{199} = \frac{99}{199} \\
P[Black \mid Jar2] &= 100 \cdot \frac{1}{199} = \frac{100}{199}
\end{aligned}$$

$$\begin{aligned}
P[Red] &= P[Red\ and\ Jar1] + P[Red\ and\ Jar2] \\
P[Red] &= P[Jar1] \cdot P[red \mid Jar1] + P[Jar2] \cdot P[Red \mid Jar2]
\end{aligned}$$

Hence,

$$\mathbf{P[Red] = (\frac{1}{2}) \cdot (1) + (\frac{1}{2}) \cdot (\frac{99}{199}) \simeq \frac{3}{4}}$$

Example 3: Repetition coding over binary symmetric channels

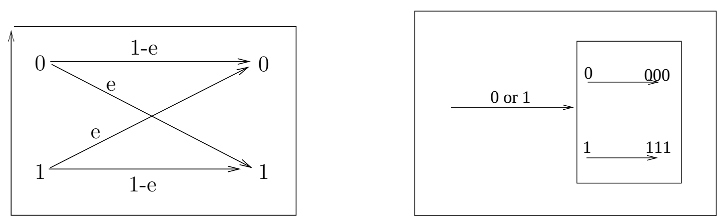

Figure 1: Repetition coding over binary symmetric channels

Suppose $\epsilon = 0.1$ Note: the e in the figure takes the value of ϵ.

In a repetition coding scheme, a bit M is repeated three times as MMM. Suppose that the received vector is ABC. (We also assume the following statement, $P[M = 1] = P[M = 0] = \frac{1}{2}$)

The decoder of the receiver calculates the number of ones in vector ABC. Denote this number by N. If $N > 1.5$, declare $M = 1$, otherwise declare $M = 0$.

What is the probability that the decoder makes an error ?

Solution: The probability for the decoder to make an error for the condition that $M = 0$ is:

$$\begin{aligned} P[error \mid M=0] &= P[Mf=1 \mid M=0] \\ &= P[N \geq 2 \mid M=0] \\ &= P[N \geq 2 \mid MMM=000] \\ &= P[\underline{ABC} \in \{\underline{011}, \underline{101}, \underline{110}, \underline{111}\} \mid MMM=000] \\ &= P[ABC=011 \mid MMM=000] + P[ABC=101 \mid MMM=000] \\ &+ P[ABC=110 \mid MMM=000] + P[ABC=111 \mid MMM=000] \end{aligned}$$

Let's start with $P[ABC = 011 \mid MMM = 000]$. If we assume that the reception of each bit are independent among each other, then:

$$P[ABC=011 \mid MMM=000] = (0.9)\cdot(0.1)\cdot(0.1) = 0.009$$
$$P[ABC=101 \mid MMM=000] = (0.1)\cdot(0.9)\cdot(0.1) = 0.009$$
$$P[ABC=110 \mid MMM=000] = (0.1)\cdot(0.1)\cdot(0.9) = 0.009$$
$$P[ABC=111 \mid MMM=000] = (0.1)\cdot(0.1)\cdot(0.1) = 0.001$$

$$\mathbf{P[error \mid M=0] = 2.8\%}$$

Similarly, we can show the following: $P[error \mid M = 1] = 2.8\%$

$$\begin{aligned} P[error] &= P[error\ and\ M=0] + P[error \mid M=1] \\ &= P[error \mid M=0]\cdot P[M=0] + P[error \mid M=1]\cdot P[M=1] \\ &= (2.8)\cdot\frac{1}{2} + (2.8)\cdot\frac{1}{2} = 2.8\% \end{aligned}$$

Note: We are not talking about the joint event of transmitting a 1 and receiving a 0, but about the conditional event of receiving a 0 given a 1 is transmitted.

The Bernoulli Trial [2]

A Bernoulli trial is a random experiment with $|S| = 2$. We refer to the two elements of a Bernoulli trial as "success" and "failure" respectively. The probability law of a Bernoulli trial is completely specified by a single number P, the probability of success.

[2] 2.6 Sequential experiments, P. 61 of the textbook

Binomial Probability Law[3]

Setup:

Perform a Bernoulli trial n times independently, with a given P. Record the number of successes, K. The sample space $S = \{0, 1, 2, ..., n\}$

$$\begin{aligned}
P[K=0] &= (1-P) \times (1-P) \times ... \times (1-P) \text{ (Performed } n \text{ times)} \\
&= (1-P)^n \\
P[K=1] &= (n \times P) \times (1-P)^{(n-1)} \\
P[K=2] &= \binom{n}{2} \times p^2 \times (1-P)^{(n-2)} \\
P[K=m] &= \binom{n}{m} \times p^m \times (1-P)^{(n-m)}
\end{aligned}$$

Multinomial Probability Law[4]

Setup:

Let an experiment have a sample space and a probability law of S and P[{m}], respectively,

$$S = \{1, 2, 3, ..., M\} \qquad P[\{m\}] = P_m, \text{ for any m} \in \{1, 2, ..., M\}$$

in such way that $P_1 + P_2 + ... + P_M = 1$.

Perform this experiment independently n times. Record $(k_1, k_2, ..., k_m)$ where k_i is the number of times i has occured $(i = 1, 2, ..., M)$. (Clearly, we can see that $k_1 + k_2 + ... + k_m = n$.)

The Sample Space S is the set of all $(k_1, k_2, ..., k_m)$ where each k_i is a non-negative integer and $k_1 + k_2 + ... + k_m = n$. What is the probability of an event $\{k_1, k_2, ..., k_m\}$? Let's look at an example.

Example 4

$S = \{1, 2, 3\}$ with (P_1, P_2, P_3) given.

$$\begin{aligned}
P[\{(K_1, K_2, K_3)\}] &= P_1^{K_1} \times P_2^{K_2} \times P_3^{K_3} \times \binom{n}{K_1} \times \binom{n-K_1}{K_2} \times \binom{K_3}{K_3} \\
&= P_1^{K_1} \times P_2^{K_2} \times P_3^{K_3} \times \frac{n!}{(n-K_1)! \cdot K_1!} \times \frac{(n-k_1)!}{K_3! \cdot K_2!} \\
&= (P_1^{K_1} \times P_2^{K_2} \times P_3^{K_3}) \times \frac{n!}{K_1! \cdot K_2! \cdot K_3!}
\end{aligned}$$

[3] 2.6 Sequential experiments, P. 61 of the textbook
[4] 2.6 Sequential experiments, P. 65 of the textbook

In general, when $|S| = M$, we have

$$P[\{K_1, K_2, ..., K_m\}] = (P_1^{K_1} \times P_2^{K_2} \times ... \times P_m^{K_m}) \times \frac{n!}{(K_1!) \cdot (K_2!) \cdot ... \cdot (K_m!)}$$