Lectures of May 12 2006 - AM
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1 Independence of Events and Experiments

Recall from last lecture:

From last example, we can note some definitions that are used for cancer tests such as:

Sensitivity: used to express how good a test can detect cancer given that a person has cancer X.
Sensitivity= $P\left['' +'' /X\right]$

Specificity: used to express how good a test can detect that a person does not have cancer given that the person does not really have cancer X.
Specificity= $P[''-''/X^c]$

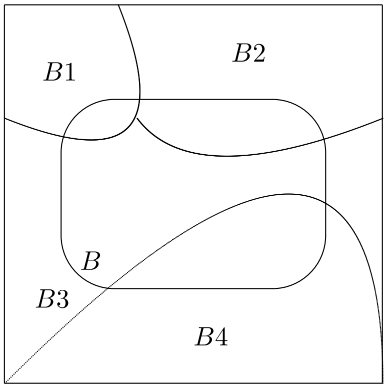
False negative rate: used to express the chances that the test would not detect cancer given that the person tested has cancer X.
False negative rate= $P[''-''/X]$

False positive rate: used to express the chances that the test would detect cancer in a person given that the tested person does not have cancer X.
False positive rate= $P[''+''/X^c]$

Using the previously proven identity $P[A/B] + P[A^c/B] = 1$

We can conclude the following:
Sensitivity + false negative rate = 1.
Specificity + false positive rate = 1.

We also recall that if a sample space S and a set of events $B_1, B_2.....B_n$ are defined such that S = $B_1 \cup B_2 \cup ..... \cup B_n$,
$\sum P[B_i/B] = 1$ as the figure below illustrates.



1.1 Independent Events

Events A and B are said to be independent if $P[A \cap B] = P[A] \cdot P[B]$.

Example

"Card-drawing" experiment:
Pick a card from a regular deck.
Let A be the event that a drawn card is red.
Let B the event that the card is an Ace.
Are A and B independent?

Solution: $P[A \cap B] = 2 \cdot \frac{1}{52} = \frac{1}{26}$

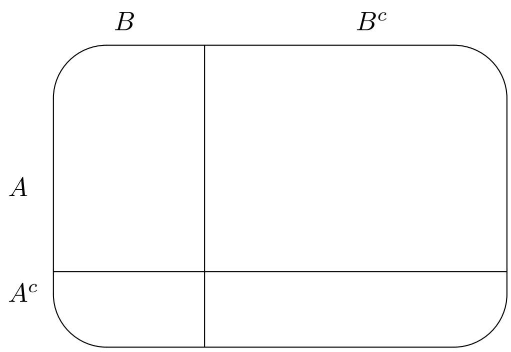
$P[A] = 26 \cdot \frac{1}{52} = \frac{1}{2}$

$P[B] = 4 \cdot \frac{1}{52} = \frac{1}{13}$

$P[A \cap B] = P[A] \cdot P[B]$

Therefore A and B are independent.

Consider sample space S with events A and its complement and B and its complement



If A and B are independent, it's easy to show that:
$P[A/B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A] \cdot P[B]}{P[B]} = P[A] \qquad (if P[B] \neq 0)$

$P[B/A] = P[B] \qquad (if P[A] \neq 0)$
$P[A/B^c] = P[A]$

Proof:
$P[A/B^c] = \frac{P[A \cap B^c]}{P[B^c]} = \frac{P[A] - P[A \cap B]}{P[B^c]} = \frac{P[A] - P[A] \cdot P[B]}{P[B^c]} = \frac{P[A](1 - P[B])}{P[B^c]} = P[A]$

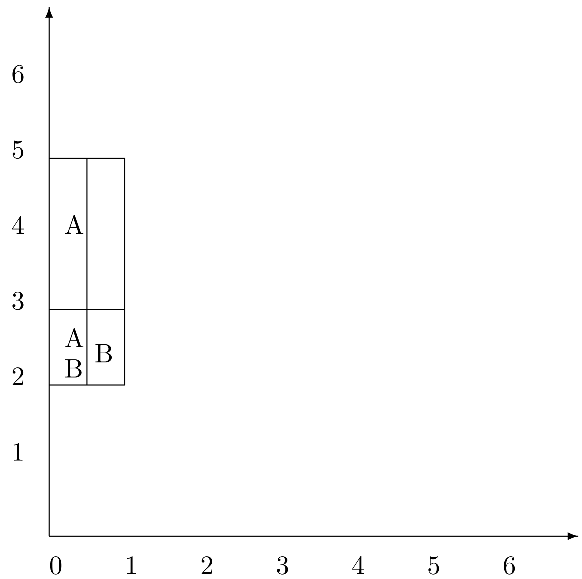
From this, we see that
$P[A \cap B^c] = P[B^c] \cdot P[A]$
$\Rightarrow$ A and B^c are independent

Example:

First pick a number X uniformly randomly from interval $(0, 1)$, and pick another number Y uniformly randomly from interval $(2, 5)$

$A = [X \leq \frac{1}{2}]$
$B = [Y \leq 3]$
$C = [X + Y \geq 3]$

Which pair(s) of events are independent?



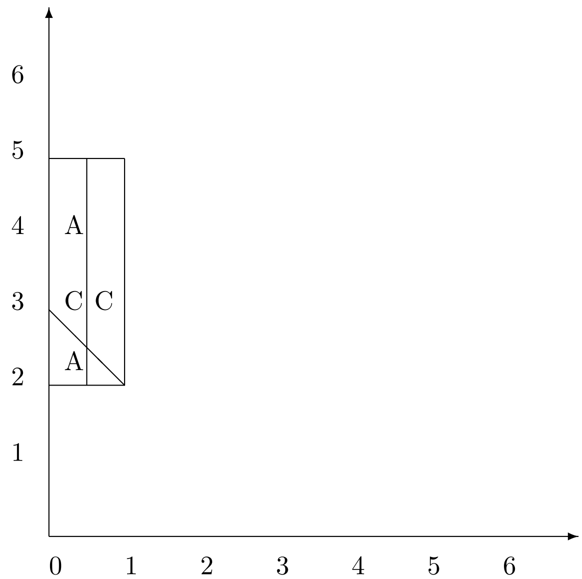
$P[A \cap B] = \frac{1}{2} \cdot \frac{1}{1 \cdot 3} = \frac{1}{6}$

$P[A] = \frac{1}{2}$

$P[B] = \frac{1}{3}$

$P[A \cap B] = P[A] \cdot P[B]$
$\Rightarrow$ A and B are independent

This result agrees with our expectations because event A operates on some interval of X, whose limits do not depend on the value of Y, and event B operates on some interval of Y, whose limits do not depend on the value of X.



$P[C] = \frac{5}{6}$
$P[A \cap C] = \frac{1.125}{3}$
$P[A] \cdot P[C] = \frac{5}{12} \neq P[A \cap C]$

Therefore A and C are <u>not</u> independent.

This result was also expected because the value of X will definitely affect the result of X + Y. So the probability that event C occurs depends on the probability that event A occurs.

1.2 Independence of Multiple Events

Events $A_1, A_2.....A_n$ are said to be independent if for any integer k $\geq$ 2 and any choice of $i_1, i_2....i_k$ such that $1 \leq i_1 \lhd i_2 \lhd i_3...... \lhd i_k \leq$ n,
$P[A_{i1} \cap A_{i2} \cap ..... \cap A_{ik}] = P[A_{i1}] \cdot P[A_{i2}] \cdot .... \cdot P[A_n]$
For example, events A, B, and C are said to be independent if
$P[A \cap B] = P[A] \cdot P[B]$
$P[A \cap C] = P[A] \cdot P[C]$
$P[B \cap C] = P[B] \cdot P[C]$
$P[A \cap B \cap C] = P[A] \cdot P[B] \cdot P[C]$

<u>Example</u>

Pick a number X from set S $\{0, 1\}$ at random where each number is picked with equal likelyhood, then pick a number Y from set S $\{0, 1\}$ where each number is picked with equal likelyhood.

Compute $Z := (X + Y) mod 2$
Record (X,Y,Z).

The sample space $S := \{000, 011, 101, 110\}$
Check whether event A := $[X = 1]$, B := $[Y = 1]$ and C := $[Z = 1]$ are independent.

$P[A] = 2 \cdot \frac{1}{4} = \frac{1}{2}$

$P[B] = 2 \cdot \frac{1}{4} = \frac{1}{2}$

$P[C] = 2 \cdot \frac{1}{4} = \frac{1}{2}$

$P[A \cap B] = P[110] = \frac{1}{4}$

$P[A \cap C] = P[101] = \frac{1}{4}$

$P[B \cap C] = P[011] = \frac{1}{4}$

$P[A \cap B] = P[A] \cdot P[B]$
$P[B \cap C] = P[B] \cdot P[C]$
$P[A \cap C] = P[A] \cdot P[C]$
$P[A \cap B \cap C] = P[111] = P[\Phi] = 0$

But $P[A] \cdot P[B] \cdot P[C] = \frac{1}{8}$

Therefore A, B, and C are <u>not</u> independent; consistent with our intuition.
But note: A and B are independent, B and C are independent, and A and C are independent.
By just looking at the elements of S, we see that the value of the first bit X alone does not tell us whether the third bit Z is a 1 or 0. Also the value of the second bit Y alone does not tell us whether the third bit Z is a 1 or 0. But we can see that Z = X xor Y. That is why the three events together are dependent.
We conclude that a set of events may not be totally independent eventhough it is pairwise independent.

1.3 Independence of Experiments

Independence of experiments are typical assumptions for a sequence of experiments that are not related in any way.
Suppose we perform a sequence of experiments $E_1, E_2, ...E_n$, and suppose these experiments are independent.
Let $S_1, S_2, ...S_n$ be the sample space of $E_1, E_2, ...E_n$. The assumption that these experiments are independent is understood as follows:
For any sequence of events $A_1, A_2, ...A_n$ with $A_i \subseteq S_i$,
$P[A_1 \times A_2.......A_n] = P[A_1 \times S_2 \times S_3.....S_n] \cdot P[S_1 \times A_2 \times S_3....S_n] \cdot P[S_1 \times S_2 \times$

$A_3 .... \times S_n] \cdot .... \cdot P[S_1 \times S_2 \times .... \times S_{n-1} \times A_n]$

Note that $P[A_i \times S_{i+1} \times ...... \times S_n]$ is the probabilty that event A_i happens in Experiment E_i regardless of the outcomes of the other experiments because this cross product gives us all patterns of the combination of A_i with the outcomes of all other experiments.

Aside Notation: For any 2 sets A and B, the cartesian product $A \times B$ is the set $\{(a, b) : a \in A, b \in \text{B}\}$

<u>Example</u>

Pick a card from a deck and put it back. Do it again. Record the pair of cards (with order).

$P[A, 3] = \frac{4\times 4}{52\times 52} = \frac{1}{169}$

where $P[A, 3]$ is the probability of the first card being an Ace and the second being a 3

$P[A, anything] = \frac{4\times 52}{52\times 52} = \frac{1}{13}$

$P[anything, 3] = \frac{52\times 4}{52\times 52} = \frac{1}{13}$

$\Rightarrow P[A, 3] = P[A, anything] \cdot P[anything, 3]$