Lectures of May 9, 2006
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Combinatories basics

Case 1: Sampling with replacement and with order

Case 2: Sampling without replacement and with order

There are $n(n-1)(n-2)...(n-k+1) = \frac{n!}{(n-k)!}$ many ways to draw. A special case in this setting is when $n = k$. The answer is $k!$.

Case 3: Sampling without replacement and without order

Setup: Pick k objects from a set of n distinct objects and record the set (i.e. without considering the order of the objects picked). There are

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

many ways.

This can be seen by first considering the case of sampling a set of n objects without replacement and with order. We know from earlier discussion that there are $\frac{n!}{(n-k)!}$ many different k-tuple. Each k-tuple in this set of configurations have $k!$ many different permutations (counting itself). Therefore, ignoring orders, there are $\frac{n!}{(n-k)!k!}$ many different "compositions".

Case 4: Sampling with replacement and without order

Setup: Pick an object from a set of n distinct objects and put it back. Do this k times and record each distinct object drawn and the number of times it is drawn.

Example 1 $n = 4$ *and* $k = 5$. *An outcome of such an experiment can be presented as the following:*

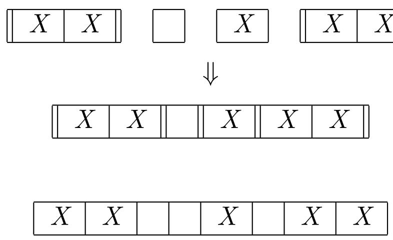
Object 1	*Object 2*	*Object 3*	*Object 4*
2	*0*	*1*	*2*

We need to solve for the number of different outcomes.

Now write the table in a different way.

Object 1	*Object 2*	*Object 3*	*Object 4*
XX		*X*	*XX*

Draw this table as 4 bins with balls



Where the blank cells are walls.

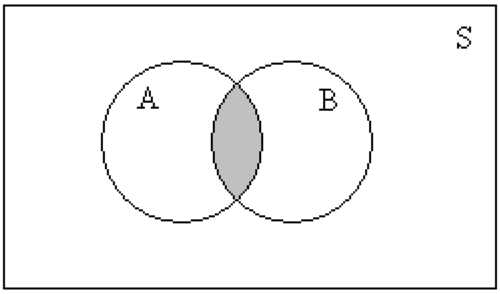
In this representation, we create $4+5-1=8$ *cells. We will fill five of them as balls and fill the remaining ones with the walls. That is, effectively we are choosing five different cells from a set of eight cells to fill balls. Therefore there are* $\binom{n}{k}$ *different ways to do so. Therefore, for the general setting of* n *and* k *we have* $\binom{n+k-1}{k}$ *different ways.*

Conditional Probability

Given a sample space S and given a probability assigment on S, we know we can calculate the probability of any event $A \subseteq S$.

Suppose that A and B are two events in sample space S, namely $A \subseteq S$ and $B \subseteq S$. We define the conditional probability of event A given event B (also referred to as the probability of A given conditionnal upon B) as

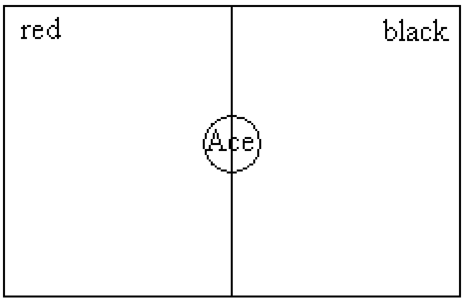
$$P[A|B] := \frac{P[A \cap B]}{P[B]}$$



Clearly by this definition

$$P[B|B] = \frac{P[B \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1$$

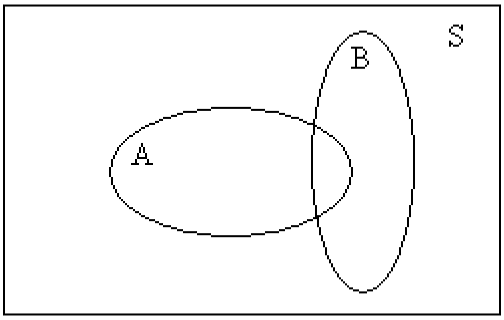
Example 2 *Pick a card from a regular deck.*



Assume each card is drawn equally likely. What is the possibility that an Ace is picked given it is red n?

$$\begin{aligned} P[ace\ picked|red\ card\ picked] &= \frac{P[red\ ace\ picked]}{P[red\ card\ picked]} \\ &= \frac{2 \times \frac{1}{52}}{26 \times \frac{1}{52}} \\ &= \frac{1}{13} \end{aligned}$$

Total Probability



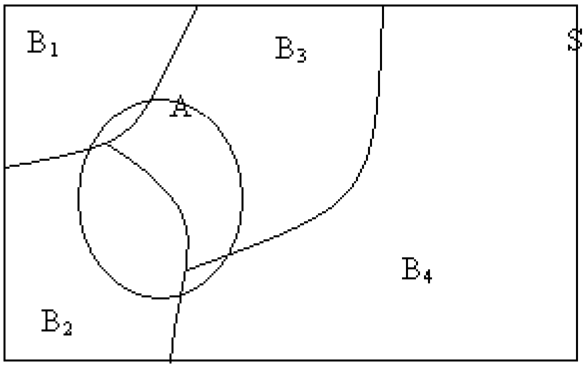
Suppose that a sample space S and a probability assigment on S is given. Let A and B be two different events.

$$\begin{aligned} P[A] &= P[A \cap B] + P[A \cap B^c] \\ &= P[B]P[A|B] + P[B^c]P[A|B^c] \\ &= P[A]P[B|A] + P[A]P[B^c|A] \end{aligned}$$

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \Rightarrow P[A|B] + P[A^c|B] = 1$$

Theorem (Total Probability)

Let A and B_1, B_2... be some events in a given sample space S with a probability law specified. Further more, B_1, B_2, ... are mutually disjoint (mutually exclusive) and $\bigcup_{\infty}^{n=1} B_n = S$.



then

$$
\begin{aligned}
P[A] &= P[A\cap\ B_1] + P[A\cap\ B_2] + \ldots \\
&= P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \ldots \\
&= \sum_{i=1}^{\infty} P[B_i]P[A|B_i]
\end{aligned}
$$

Theorem

$$
P[A|B] = \frac{P[A\cap B]}{P[B]} = \frac{P[B|A]P[A]}{P[B]}
$$

Theorem (Bayes Rule)

Let B_1, B_2, ... be mutually disjoint and $\bigcup_{i=1}^{infty} = S$ then for any $A \subseteq S$

$$
P[B_i|A] = \frac{P[A|B_i]P[B_i]}{\sum_{n=1}^{\infty} P[A|B_i]P[B_i]}
$$

Example 3 *Suppose that there is a test for cancer X. Suppose the probability of test positive is* 0.2 *(this may be determined by the fraction of the population having test positive) (this means the probability of test negative is* 0.8*). It is known that the possibility of having cancer X given test positive is 0.8 (this may be determined by the fraction of cancer X patients in the test positive population). Is is also known that the probability of having cancer X given test negative is* 0.1*. Suppose John takes the test, what is the probability the test is positive? What is the probability John has cancer X?*

Answer:

$$P[X|+] = 0.8$$

What is the probability that John has cancer without taking the test?

Answer:

$$\begin{aligned} P[X] &= P[X \cap +] + P[X \cap -] \\ &= P[+]P[X|+] + P[X|-]P[-] \\ &= 0.2 \times 0.8 + 0.1 \times (1 - 0.2) \\ &= 0.16 + 0.08 \\ &= 0.24 \end{aligned}$$

What is the probability of test positive given one has cancer X?

Answer:

$$\begin{aligned} P[+|X] &= \frac{P[+ \cap X]}{P[X]} \\ &= \frac{P[X|+]P[+]}{P[X]} \\ &= \frac{0.8 \times 0.2}{0.24} \\ &= \frac{2}{3} \end{aligned}$$