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1 Random experiment, outcome, Sample Space and events:

A Random experiment: is specified by an experimental procedure and a set of outcomes (measurements / observations etc).
The set of all of all outcomes is called Sample Space, which we will often denote by $\mathcal{S}$.

Example 1 *A "Card- Drawing" experiment may be specified as follows:*

1. *Pick a card from a deck of regular cards (52 cards).*
2. *Record the number and the Suit of the Card.*

The Sample Space $\mathcal{S}$ of this Random experiment is:

$$\mathcal{S} = \{1\heartsuit, 2\heartsuit, ..., K\heartsuit, 1\clubsuit, 2\clubsuit, ..., K\clubsuit, 1\diamondsuit, 2\diamondsuit, ..., K\diamondsuit, 1\spadesuit, 2\spadesuit, ..., K\spadesuit\}$$

Any subset of the Sample Space is called an event. The Sample Space $\mathcal{S}$ itself is by definition a subset of $\mathcal{S}$ and is therefore an event sometime called" Certain event". The empty set ϕ is also a subset of the Sample Space; sometime it is called the "Impossible event".Any subset of $\mathcal{S}$ that contains only one element is called" an elementary event".

Example 2 *Let an experiment be defined as follows:*

1. *Pick a real number $\boldsymbol{X}$ at random from interval $[-1, 1]$ then pick a real number $\boldsymbol{Y}$ at random from interval $\left[0, \boldsymbol{X}^2\right]$.*
2. *Note the pair $(\boldsymbol{X}, \boldsymbol{Y})$.*

The Sample Space:

$$\mathcal{S} = \{(\boldsymbol{X}, \boldsymbol{Y}) : -1 \leq \boldsymbol{X} \leq 1, 0 \leq \boldsymbol{Y} \leq \boldsymbol{X}^2\}$$

The event $\{|\boldsymbol{X}| \leq 0.5\}$ in this Sample Space is the area which can be expressed as $\{(\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{S} : |X| \leq 0.5\}$.

Example 3 *An experiment is defined as follows:*

1. *Keep tossing a coin until 2 "'H"' are seen.*
2. *Record the Sequence of tossing results.*

The Sample Space $\mathcal{S}$ at this experiment is the set of all sequences each containing two "'H"' and ending with "'H"'.

Axioms of Probability

A probability Law of a Random Experiment is a rule that assigns every event $\mathcal{A}$ in the Sample Space $\mathcal{S}$ of the experiment a number, which will be denoted by $\mathcal{P}(\mathcal{A})$.and referred to as the Probability of $\mathcal{A}$, such that the following axioms are satisfied:

1. For any set $\mathcal{A} \subseteq \mathcal{S}, \mathcal{P}(\mathcal{A}) \geq 0$.
2. $\mathcal{P}(\mathcal{S}) = 1$.
3. For any sequence $\mathcal{A}_1, \mathcal{A}_2, ...$ of events, if they are disjoint (meaning $\mathcal{A}_i \cap \mathcal{A}_j = \phi$, for any $i \neq j$), then:
$$\mathcal{P}(\bigcup_{i=1}^{n} \mathcal{A}_i) = \sum_{i=1}^{n} \mathcal{P}(\mathcal{A}_i)$$
where $\bigcup_{i=1}^{n} \mathcal{A}_i = \mathcal{A}_1 \cup \mathcal{A}_2 \cup ... \cup \mathcal{A}_n$.

Properties of Probability Law

Corollary 1 *For any set* $\mathcal{A} \subseteq \mathcal{S}$, $\mathcal{P}(\mathcal{A}) = 1 - \mathcal{P}(\mathcal{A}^c)$.

Proof:

$\mathcal{A}$ *and* $\mathcal{A}^c$ *are disjoint by Axiom 3, then:*

$$\mathcal{P}(\mathcal{S}) = \mathcal{P}(\mathcal{A} \cap \mathcal{A}^c) = \mathcal{P}(\mathcal{A}) + \mathcal{P}(\mathcal{A}^c)$$

By Axiom 2, $\mathcal{P}(\mathcal{S}) = 1$*, therefore,* $\mathcal{P}(\mathcal{A}) = 1 - \mathcal{P}(\mathcal{A})^c$

Corollary 2 $\mathcal{P} \leq 1$*, for any* $\mathcal{A} \subseteq \mathcal{S}$.

Proof:

From corollary 1, $\mathcal{P}(\mathcal{A}) = 1 - \mathcal{P}(\mathcal{A})^c$*. By Axiom 1,* $\mathcal{P}(\mathcal{A}^c) \geq 0 \Rightarrow \mathcal{P}(\mathcal{A}) \leq 1$.

Corollary 3 $\mathcal{P}(\phi) = 0$

Proof:

$\mathcal{S} = \mathcal{S} \cup \phi$, $\mathcal{S}$ *and* ϕ *are disjoint.*

$\mathcal{P}(\mathcal{S}) = \mathcal{P}(\mathcal{S}) + \mathcal{P}(\phi)$*, (Axiom 3).*

$\mathcal{P}(\mathcal{S}) = 1$*, we have* $\mathcal{P}(\phi) = 0$.

Corollary 4 $\mathcal{P}(\mathcal{A} \cup \mathcal{B}) = \mathcal{P}(\mathcal{A}) + \mathcal{P}(\mathcal{B}) - \mathcal{P}(\mathcal{A} \cap \mathcal{B})$ *for any* $\mathcal{A} \subseteq \mathcal{S}$*, for any* $\mathcal{A} \subseteq \mathcal{S}$.

Corollary 5 *if* $\mathcal{A} \subseteq \mathcal{B}$*, then* $\mathcal{P}(\mathcal{A}) \leq \mathcal{P}(\mathcal{B})$.

(Assignment of Probability (specifying a Probability Law)

We have noted from previous examples that Sample Space can be finite, countably infinite and unaccountably infinite for all different uses the assignments of Probability can be different.

The general rule to assign probabilities is to make the three Axioms satisfied.

Case 1 *$\mathcal{S}$ is infinite*

Option 1 *(with least assumption)*

Equally-likelihood assignment. Suppose $|\mathcal{S}| = m$, (i.e. $\mathcal{S}$ has m elements), using equal-likelihood assignment we will assign $\mathcal{P}(\mathcal{A}) = \frac{1}{m}$ for any $\mathcal{A} \in \mathcal{S}$.

Example 4 *Coing tossing*

Equal-likelihood assignemt: $\mathcal{P}(H) = \mathcal{P}(T) = \frac{1}{2}$.

A non-equal-likelihood assignment can be: $\mathcal{P}(H) = \frac{1}{3}$, $\mathcal{P}(T) = \frac{2}{3}$

Example 5 *Toss three coins and note the outcomes as a (triple).*

The sample space $\mathcal{S} = \{HHH, HHT, HTH, THH, THT, TTH, TTT\}$

Under equal likely assignment $\mathcal{P}(HHH) = \mathcal{P}(HHT) = ... = \mathcal{P}(TTT) = \frac{1}{8}$.

Example 6 *Toss three coins and note the number of heads. The sample space $\mathcal{S} = \{0, 1, 2, 3\}$.*

From the previouse example, we see that equal-likelihood assignment here is "'not reasonable"'.

We assign probability based on the previouse exmaple:

$$\mathcal{P}(\{0\}) = \mathcal{P}(\{3\}) = \frac{1}{8}, \quad \mathcal{P}(\{1\}) = \mathcal{P}(\{2\}) = \frac{3}{8}$$