ELG3121 Review Questions

July 11, 2005

Question 1 *The pdf of random variable X is given in Figure 1, where the height of the central triangle is not given.*

1. Determine the height of the central triangle.
2. Find the cdf of X.
3. Find the mean of X.
4. Find the variance of X.

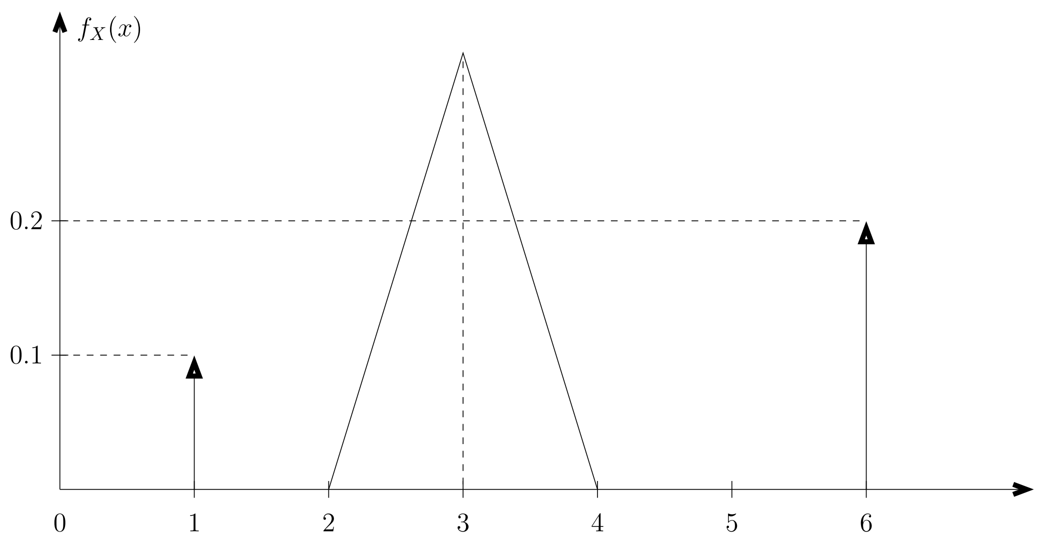

Figure 1: The probability density function of random variable X.

Question 2 *The pdf of random variable Y is given in Figure 2. Suppose that random variable Y is independent of random variable X whose pdf is given in Figure 1. Let random variable Z be defined as $Z := X + Y$.*

Determine the pdf of Z.

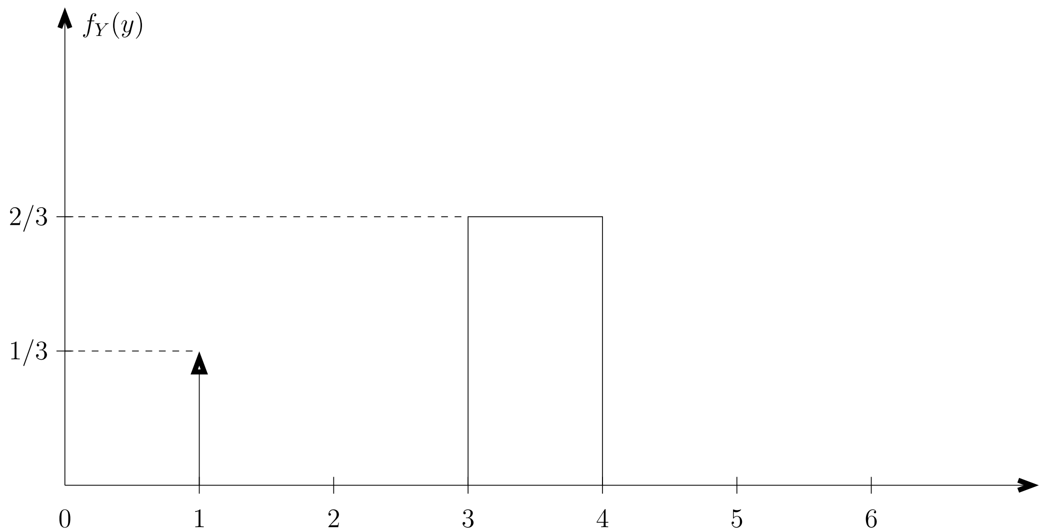

Figure 2: The probability density function of random variable Y.

Question 3 *John is going to do the following experiment. He will roll a fair die; if the outcome is* 1*, he will pick a number at random from distribution* $f_X(x)$ *in Figure 1; and if the outcome is not* 1*, he will pick a number at random from distribution* $f_Y(y)$ *in Figure 2. Denote by* W *the random number John picks.*

Determine the distribution of W.

Question 4 *Suppose that subroutine* `rand()` *implemented in some programming language returns a random number drawn from the uniform distribution on interval* $(0, 1)$*. Now except for calling subroutine* `rand()`*, you are only allowed to use* `if` *statements, assignments (i.e., statement like* `a=0.5`*), and addition operations.*

Describe a procedure (preferably in a pseudo-code) that implements random variable X whose pdf is given in Figure 1.

Question 5 *Random variables* T*,* U *and* V *are all binary, namely, taking values from* $\{0, 1\}$*. Suppose that* T *and* U *are independent and both Bernoulli with* $p = 1/2$*. Random variable* V *is defined as*

$$V := T + U \mod 2.$$

1. Determined the joint distribution of T, U, and V.
2. Are T and V independent?
3. Are T, U and V independent?
4. Suppose that we observe $U = 0$. Given this observation, are T and V independent?

Question 6 *Let function $h(x)$ be defined by $h(x) := 2x - 1$. Vector random variable (T, U, V) in Question 5 are converted to another vector random variable $(h(T), h(U), h(V))$. Suppose that this vector (essentially the electrical signals representing (T, U, V)) is transmitted through some noisy channel, and the received vector is modelled as $(h(T) + N_1, h(U) + N_2, h(V) + N_3)$, where N_1, N_2, N_3 are independent Gaussian random variables each having mean 0 and variance σ^2. We note that (N_1, N_2, N_3) are independent of $T, U,$ and V respectively.*

1. Determine the covariance $COV(h(T) + N_1, h(U) + N_2)$.
2. Suppose at the receiver, $(h(T) + N_1, h(U) + N_2, h(V) + N_3)$ is observed as $(5, -3, -2)$. Given this observation, what is the most likely vector configuration for (T, U, V)?

Question 7 *Vector random variable (T, U, V) in Question 5 are observed by John. John generates three random i.i.d Bernoulli random variables (B_1, B_2, B_3) where each $B_i, i = 1, 2, 3$, has mean 0.1. He then generates another random vector $(T \oplus B_1, U \oplus B_2, V \oplus B_3)$, where $a \oplus b := a + b \mod 2$.*

Suppose you observe $(T \oplus B_1, U \oplus B_2, V \oplus B_3)$ as $(1, 1, 1)$ but did not observe (T, U, V), what is the most likely vector configuration for (T, U, V) that John observed?

Question 8 *Suppose that random variable S is defined via random variable X in Figure 1 as*

$$S := \begin{cases} X^3 + 5, & \text{if } X \geq 3 \\ 32, & \text{otherwise.} \end{cases}$$

Determine the pdf of S.

Question 9 *Suppose that $X_1, X_2, \ldots, X_{50}$ are 50 i.i.d Gaussian random variables with unknown mean μ, and variance 25. Let $M := (X_1 + X_2 + \ldots X_{50})/50$.*

1. Express the pdf of M, where μ can be a parameter.
2. Determine the probability that M is in the interval $[\mu - 1, \mu + 1]$, in terms of Q function.

Question 10 *Roll a fair die repeatedly and suppose that each trial is independent of all other trials. A trial is said to have a low outcome if the outcome is 1 or 2, to have a median outcome if the outcome is 3 or 4, and to have a high outcome if the outcome is 5 or 6.*

1. Determine the probability that the number of high outcomes in the first 50 trials equals 30.
2. Determine the probability that the number of high outcomes in the second 50 trials equals 30 given that the number of high outcomes in the first 50 trials is 0.
3. Determine the probability that trial 5 is the first trial in which a low outcome occurs.
4. Determine the probability that trial 5 is the first trial in which a low outcome does not occur.

5. Suppose that after 10 trials, we observe precisely 3 low outcomes, determine the probability that the fourth low outcome occurs in trial 15.

6. Let X_1 and X_2 denote the outcomes (in terms of dots) for trial 1 and trial 2 respectively. Suppose that you are told $X_1 < X_2$, determine the joint pmf of X_1 and X_2 given this knowledge.

Question 11 *A man has two umbrellas. Every day he leaves home for his office in the morning, and comes back home in the evening. In the morning, he will take an umbrella with him to go to work if and only if*

1. *it rains and*

2. *he has an umbrella at home.*

Similarly, in the evening, he will take an umbrella with him to go home if and only if

1. *it rains and*

2. *he has an umbrella in his office.*

The probabilities of raining in the morning and in the evening are both 0.1*. It is also assumed that raining in the morning and raining in the evening are independent events; further, across days, whether it rains in the morning or evening is also independent.*

On the morning of day 1 before he leaves home, the man has two umbrellas at home.

1. What is the probability that he will get wet in the evening of day 1 (i.e., going back home without umbrella while it is raining)?

2. Denote by X_i the number of umbrellas the man has at home in the evening of day i after he comes back from work. Find the pmf for X_2.

3. Find the pmf of X_3.

Question 12 *Random variables X and Z are independently Gaussian with means $\mu_X = 5, \mu_Z = 3$ and variances $\sigma_X^2 = 16$ and $\sigma_Z^2 = 4$. Random variable Y is defined as $Y := aX + Z$.*

1. Find the mean of Y.

2. Find the covariance and correlation coefficient between X and Y.

3. Find the joint pdf f_{XY}.

4. For which value of a, X and Y are independent?

Question 13 *Verify which of the following claims are correct (if they are not correct, you should try to give a counter-example).*

1. If X and Y are independent, then they are uncorrelated.

2. If X and Y are uncorrelated and both have zero mean, then they are independent.

3. If X and Y are orthogonal and one of the two variables has zero mean, then X and Y are uncorrelated.

4. If X and Y are independent, Y and Z are independent, then X and Z are independent.

5. If X and Y are independent, Y and Z are independent, and X and Z are independent, then X, Y, and Z are independent.

6. $E[X+Y] = E[X] + X[Y]$.

7. $VAR[X+Y] = VAR[X] + VAR[Y]$.

8. Two jointly Gaussian random variables are independent if they are uncorrelated.

9. Two jointly Gaussian random variables are independent if they both have zero mean.

10. If X and Y are jointly Gaussian random variables, then $X+Y$ is a Gaussian random variable.

11. If X and Y are both Gaussian random variables, then (X, Y) is jointly Gaussian.

12. If X and Y are both Gaussian random variables, then $(1-\alpha)f_X(t) + \alpha f_Y(t)$ is a Gaussian pdf, for any $\alpha \in (0, 1)$.

13. If X and Y are both Bernoulli random variables, then $(1-\alpha)p_X(k) + \alpha p_Y(k)$ is a Bernoulli pmf, for any $\alpha \in (0, 1)$.

14. If the correlation coefficient $\rho_{XY} = -1$, then X and Y are uncorrelated.

15. If two events are mutually exclusive (i.e., disjoint as subsets), then they are independent.

16. Suppose that X is a Poisson random variable with mean α and that $Y = kX + b$ for some $k \neq 0$ and some real number b. Then the correlation coefficient ρ_{XY} is 1.

Question 14 *It is hypothesized that a die is biased with distribution $p(1) = p(2) = p(3) = p(4) = p(5) = 0.1$ and $p(6) = 0.5$. The die is tossed 120 times, and the number of each outcome is observed as follows.*

k	1	2	3	4	5	6
N_k	14	16	17	23	11	39

At 1% significance level, would you reject this hypothesis?

Question 15 *Random variable X takes on values from $\{2, 3, 6\}$, with distribution $p_X(2) = 1/2, p_X(3) = 1/3$, and $p_X(6) = 1/6$. Let $Y(t) = \cos(2\pi X t)$. That is, for any fixed value t, $Y(t)$ is a random variable; and when X is observed, $Y(t)$ becomes a function of t if we let t vary.*

1. Find the pmf of $Y(1/2)$.

2. Find the pmf of $Y(1)$.

3. Find the joint pmf of $Y(\frac{1}{2})$ and $Y(1/3)$.

4. Find the covariance between $Y(1/2)$ and $Y(1/3)$.

Question 16 *We have calculated in class that there are about* $14,000,000$ *different combinations in 649 lottery. Suppose that statistics show that in each 649 draw, on average there are 1.5 person winning the Jackpot. As a rule, if more than one person wins the Jackpot, the winners will split the Jackpot prize. Now this weekend 649 has Jackpot valued* $\$30,000,000$, *and we will assume that this is the only prize that one can win from 649 (i.e., we will ignore smaller prizes). Each 649 ticket costs one dollar.*

Suppose that you will buy all the combinations of 649 for this weekend. Give an estimate of the probability that you will not lose money, based on only the information given in the question. — You will need to make modelling assumption and justify why your model makes sense.

Question 17 *Denote by* $\mathcal{C}$ *the region* $\{(x,y) : x^2 + y^2 \leq 4\}$. *Pick a point* (X, Y) *at random from* $\mathcal{C}$ *where each point of* $\mathcal{C}$ *is equally likely to be selected.*

1. Find the distribution of X.

2. Find the distribution of X conditioned on that (X, Y) is not in the interior of $\mathcal{C}$ and $Y = 1$.

3. Are X and Y independent?

4. Are X and Y independent conditioned on that $|X| < 1$ and $|Y| < 1$?

Question 18 *Let the inter-arrival time of messages be modelled as an exponential random variable with mean* 2.

1. Given that no message has arrived until time 6, determine the expected value of the arrival time of the first message.

2. Given that 5 messages have arrived until time 6, determine the expected value of the number of arrived messages until time 10 .