Robustness of Massive MIMO to Location and Phase Errors

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***Abstract*—The impact of random errors in element locations and beamforming phases on the performance of massive multiple-input multiple-output (MIMO) systems and their ability to cancel inter-user interference (IUI) are studied. For an arbitrary array geometry, user orthogonality, also known as "favorable propagation" (FP), is shown to hold asymptotically for the perturbed array as long as it holds for the unperturbed one, for independent (possibly non-Gaussian) errors. This means that small errors do not have catastrophic impact on the FP, even for a large number of antennas, and IUI can be reduced to any desired level. The negative impact of random errors is to slow down the convergence to the asymptotic value so that more antennas are needed under random errors to achieve the same low IUI as without errors. Practical design guidelines are given as to what implementation accuracy is needed to make the impact of random errors negligible and a closed-form estimate of IUI under random errors is presented. The analytical results are validated via numerical simulations and are in agreement with measurement-based studies.**

I. Introduction

Since the seminal work by Marzetta [1], massive MIMO (mMIMO) has being attracting significant and increasing attention, both in academia [2][3] and industry, especially for 5/6G applications [4][5]. Its main advantage is a significant increase in spectral and energy efficiency as well as simplified processing in multi-user environments [2]-[6]. This is due to a phenomenon known as "favorable propagation" (FP), whereby different users' channels become orthogonal (or nearly so) to each other when the number of base station antennas increases, thereby substantially reducing inter-user interference (IUI), even with simple linear processing [6], to any desired value provided the number of antennas is large enough.

The FP property, which ensures that low IUI is achievable, has been studied both theoretically [6]-[8] and experimentally [9]-[11]; antenna array geometry and wireless channel properties were identified as key factors affecting the FP. In particular, it was shown, using the law of large numbers, that the FP holds in i.i.d. fading channels [2][6]. However, the i.i.d. fading assumption neglects the impact of antenna array geometry and is justified provided that (i) multipath is rich enough (without a single dominant component) and (ii) antenna spacing is large enough. If either of these conditions is violated, e.g. if there is a dominant line-of-sight (LOS) component, then the i.i.d. assumption does not hold anymore. In fact, the LOS environment is extreme opposite of i.i.d. fading and is considered to be "particularly difficult" for users' orthogonality and mMIMO system performance [9]; the law of large numbers is not applicable in this case. Real-world channels are expected to be somewhere in-between of these 2 extremes [6][7]. mMIMO in LOS environment was studied in [6]-[8] and the FP was shown to hold for various array geometries under fairly broad conditions.

However, the above studies assume perfect channel knowledge/estimation, perfect element location or array calibration, no inaccuracies in beamforming weights etc. In practice, such perfect setting is hardly possible as implementation inaccuracies and tolerances always exist. It is not clear whether the FP property will still hold under such inaccuracies, when the number of antennas increases without bound. The robustness property, whereby small perturbations/errors in element locations, beamforming weights etc. do not have a catastrophic impact on the performance, is desirable from a practical perspective, especially for mmWave and THz systems, where the impact of implementation inaccuracies becomes more pronounced; in addition, the presence of phase noise also becomes a major issue affecting the system performance.

The impact of implementation inaccuracies/errors on the traditional antenna arrays has been studied and robust beamforming strategies have been proposed, including the well-known diagonal loading technique and its modifications [12][13]. A general conclusion is that, under proper design, small errors do not have a catastrophic impact on the performance. Similar results were established for uncertain MIMO channels as well [14]. However, the above results apply to the traditional settings (not mMIMO) and it remains unclear whether they still holds in the mMIMO setting and apply to the FP/user orthogonality as well (especially because increasing the number of antennas to very large values has a potential to "amplify" small per-element errors and generate a large aggregate effect thereby destroying the FP/user orthogonality).

This paper studies the robustness of the FP/user orthogonality and low IUI in mMIMO to random errors in element locations and beamforming phases. Both Gaussian and non-Gaussian error distributions are allowed. We show that, for an arbitrary array geometry, the FP/user orthogonality holds for the perturbed array under random independent errors as long as it holds for the nominal array (i.e. the one without errors), see Theorem 1. Based on this, a closed-form estimate of IUI power is proposed and conditions for errors to be negligible are given. We further show that while random errors do not affect the FP asymptotically, they have a profound negative impact on the convergence speed to the asymptotic value as the number N of antennas increases: while the IUI power scales as N^{-2} for the nominal array (no errors), it scales only as N^{-1} for the perturbed array, which is a similar scaling as in the i.i.d Rayleigh fading channel [7], so that more antennas are needed to achieve the same low IUI under random errors.

II. System Model

Let us begin with the standard basedband model [6] whereby a base station (BS) equipped with an N-elements

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antenna array serves M single-antenna users simultaneously:

$$\boldsymbol{y}(t)=\boldsymbol{h}_1 x_1(t)+\sum_{i=2}^{M}\boldsymbol{h}_i x_i(t)+\boldsymbol{\xi}(t) \quad (1)$$

where $\boldsymbol{y}(t)$ is the vector signal received by the BS at time t, $\boldsymbol{\xi}(t)$ is a zero-mean white Gaussian circularly-symmetric noise vector of variance σ_0^2 per dimention; $\boldsymbol{h}_i$ and $x_i(t)$ are the channel vector and transmitted signal of user i, $i=1...M$, respectively; $|\boldsymbol{h}|$, $\boldsymbol{h}'$ and $\boldsymbol{h}^+$ denote Euclidean norm (length), transposition and Hermitian conjugation, respectively.

To detect user 1 (the main user) signal $x_1(t)$, a linear beamforming is used (which is attractive due to its simplicity, robustness and suitability for parallel/distributed implimentation) [6], and the other users' contribution $\sum_{i=2}^{M}\boldsymbol{h}_i x_i(t)$ is treated as interference. The main user SINR can be expressed as follows:

$$\text{SINR}=\frac{|\boldsymbol{w}_1^+\boldsymbol{h}_1|^2\sigma_{x_1}^2}{\sum_{i=2}^{M}|\boldsymbol{w}_1^+\boldsymbol{h}_i|^2\sigma_{x_i}^2+|\boldsymbol{w}_1|^2\sigma_0^2} \quad (2)$$

$$=\frac{|\alpha_{11,N}|^2\gamma_1}{\sum_{i=2}^{M}|\alpha_{1i,N}|^2\gamma_i+1}\leq\gamma_1 \quad (3)$$

where σ_{xi}^2 and $\gamma_i=|\boldsymbol{h}_i|^2\sigma_{xi}^2/\sigma_0^2$ are the transmitted signal power and the received SNR of user i, respectively; $\boldsymbol{w}_1$ is the beamforming vector for detecting user 1 and

$$\alpha_{1i,N}=\frac{\boldsymbol{w}_1^+\boldsymbol{h}_i}{|\boldsymbol{w}_1||\boldsymbol{h}_i|} \quad (4)$$

where $|\alpha_{11,N}|^2\leq 1$ represents the normalized channel power gain of the main user; $|\alpha_{1i,N}|^2\leq 1$, $i=2...M$, is the IUI power "leakage" factor of user i to the main user. The channel is normalized so that $|\boldsymbol{h}_i|^2=N$ and the propagation path loss is absorbed into the single-user SNR γ_i.

In the case of no perturbations/errors, the channel is known precisely and the beamforming weights are also set precisely (as is usually assumed in the literature [6]-[8]). In this case and for the matched filter beamforming (also known as maximum ratio combining, which maximizes the single-user SNR), $\boldsymbol{w}_1=\boldsymbol{h}_1$, where $\boldsymbol{h}_1$ is the user 1 channel, so that $\alpha_{11,N}=1$. Under this condition, the upper bound in (3) is attained with equality and thus the SINR is maximized achieving its single-user value, if the total IUI power "leakage" vanishes, $\sum_{i=2}^{M}|\alpha_{1i,N}|^2=0$. This favorable condition can be approached, in certain scenarios, by increasing the number of antennas, which is known as (asymptotically) favorable propagation [6]-[8]. For a finite number of users and for uniformly-bounded per-user SNRs, the FP holds if

$$\lim_{N\to\infty}\sum_{i=2}^{M}|\alpha_{1i,N}|^2=0 \text{ or } \lim_{N\to\infty}|\alpha_{1i,N}|^2=0\ \forall i>1 \quad (5)$$

so that IUI becomes negligible and the SINR approaches its maximum, SINR $=\gamma_1$ (single-user value), as N increases. Note that, when the FP holds, users become orthogonal to each other and the performance of matched filtering, zero-forcing and MMSE receivers are the same.

In the case of random errors, (5) cannot be used anymore since $\alpha_{1i,N}$ becomes a random sequence and thus the limits in (5) do not exist (in the deterministic sense). Hence, an extension of the FP/user orthogonality condition in (5) is needed to accommodate random errors. This is done in the next section.

III. The impact of Location and Phase Errors

Let us consider an N-element antenna array of arbitrary geometry where element locations as well as beamforming phases are subject to random errors (perturbations). In particular, the actual location vector $\boldsymbol{p}_n$ of n-th antenna array element is

$$\boldsymbol{p}_n=\boldsymbol{p}_n^0+\Delta\boldsymbol{p}_n,\ n=1...N \quad (6)$$

where $\boldsymbol{p}_n^0$ is the nominal location vector and $\Delta\boldsymbol{p}_n$ is its random offset, all measured in wavelengths. Following [6]-[8], we consider the LOS environment since there are many important LOS application scenarios and because it presents an extreme opposite of iid Rayleigh fading and is particularly difficult for the FP (practical channels are somewhere in-between). In this case, the normalized channel vector entries for user i can be expressed as [6][12]

$$h_{in}=\exp(j2\pi\boldsymbol{u}_i^+\boldsymbol{p}_n),\ i=1..M,\quad n=1...N, \quad (7)$$

where i and n are user and element indexes, $\boldsymbol{u}_i$ is the unit direction vector for user i.

When matched filtering is used to detect user 1 signal, the beamforming weights $\boldsymbol{w}_1=[w_1,...,w_N]'$ are matched to user 1 nominal (rather than actual) channel vector $\boldsymbol{h}_1^0$, whose entries are $h_{1n}^0=\exp(j2\pi\boldsymbol{u}_1^+\boldsymbol{p}_n^0)$ and are perturbed in phase as well, typically due to imperfect phase shifters and quantization errors, so that

$$w_n=\exp(j2\pi\boldsymbol{u}_1^+\boldsymbol{p}_n^0+j\Delta\phi_n) \quad (8)$$

where $\Delta\phi_n$ are beamforming phase errors; following the widely-accepted approach [12], they are modeled as zero-mean i.i.d. random variables. Likewise, $\Delta\boldsymbol{p}_n$ are also modeled as zero-mean i.i.d. random vectors. Unlike the existing studies, we do not assume here that they are Gaussian, so non-Gaussian distributions are allowed as well.

Using (7) and (8), $\alpha_{1i,N}$ can be expressed as

$$\alpha_{1i,N}=\frac{1}{N}\sum_{n=1}^{N}e^{j\Psi_{in}},\quad \Psi_{in}=\Psi_{in}^0+\Delta\Psi_{in} \quad (9)$$

$$\Psi_{in}^0=2\pi(\boldsymbol{u}_i-\boldsymbol{u}_1)^+\boldsymbol{p}_n^0 \quad (10)$$

$$\Delta\Psi_{in}=\Delta\psi_{in}-\Delta\phi_n \quad (11)$$

where Ψ_{in}^0 represents phase differences of i-th and 1st users' signals for the nominal array; $\Delta\psi_{in}=2\pi\boldsymbol{u}_i^+\Delta\boldsymbol{p}_n$ represents extra phase shifts due to element location errors. The corresponding IUI leakage factor of the nominal array is

$$\alpha_{1i,N}^{\circ}=\frac{1}{N}\sum_{n=1}^{N}e^{j\Psi_{in}^{\circ}} \quad (12)$$

Since, under random errors above, $|\alpha_{1i,N}|^2$ is a random sequence (indexed by N), (5) cannot be used since the respective limits do not exist in the deterministic sense. Therefore, one has to use a notion of stochastic convergence. The following definition gives 3 such notions, which are widely used in many areas of stochastic analysis and applications; this is a slight extension of the definition in [15, p. 306].

Definition 1. *A sequence $z_1, z_2, \ldots$ of random variables converges to a deterministic sequence $a_1, a_2, \ldots$ in the mean square sense (mse), $z_N \xrightarrow{mse} a_N$, if*

$$\lim_{N\to\infty} \mathbb{E}\{|z_N - a_N|^2\} = 0 \quad (13)$$

z_N converges in probability to a_N, $z_N \xrightarrow{\Pr} a_N$, if, for any $\epsilon > 0$,

$$\lim_{N\to\infty} \Pr\{|z_N - a_N| > \epsilon\} = 0 \quad (14)$$

z_N converges to a_N almost surely (a.s.) or with probability one, $z_N \xrightarrow{a.s.} a_N$, if

$$\Pr\left\{\lim_{N\to\infty} (z_N - a_N) = 0\right\} = 1 \quad (15)$$

where the probability applies to the set of all events where the limit exists and equals to 0.

We will further use $\rightarrow$ (without a superscript) to denote stochastic convergence in all 3 senses above as well as the regular (deterministic) convergence when all related sequences are deterministic. Note that (13) and (15) imply (14) but the converse is not true in general [15].

A few remarks are in order as to why these 3 different modes of convergence are needed here. First, mean square error is a time-tested tool in many areas of communications, signal processing and stochastic control, including robust beamforming [12][13], so its use is appropriate here. It ensures that random IUI z_N converges to its mean value a_N in the MSE sense. Second, $\Pr\{|z_N - a_N| > \epsilon\}$ is the probability that the deviation of random IUI from its mean is not small. This is akin to outage probability widely used in may areas of wireless communications. In fact, it becomes the outage probability if one designs a system based on the mean IUI and the actual random IUI deviates significantly from this design. Lastly, (15) is needed since, even if (14) holds, it does not guarantee that $|z_N - a_N|$ cannot become arbitrary large (i.e., large deviation) for infinitely-many N [16, p. 237]. Such a guarantee, that $|z_N - a_N|$ becomes and *stays* small as N increases, is provided by (15) (the set of all events where this does not hold has a combined probability measure of zero, i.e. extremely unlikely to be encountered in the real world). It comes the closest to the deterministic convergence and guarantees that once a mMIMO design is acceptable for a given (large) number N_0 of antennas, it will also *remain* acceptable for any $N > N_0$. In fact, the results we establish below hold for all 3 modes of convergence, which also ensures that they are not an artifact of a particular definition used.

Next, we replace the deterministic limits in the FP definition in (5) by the stochastic convergence modes in Definition 1. Upon this replacement, the FP holds under random perturbations if, as $N \rightarrow \infty$,

$$|\alpha_{1i,N}|^2 \rightarrow 0 \ \forall i > 1 \quad (16)$$

The following Theorem establishes the FP property (16) for a perturbed array of arbitrary geometry for all 3 convergence modes above.

Theorem 1. *Under i.i.d. Gaussian perturbations, the FP holds for a perturbed array defined above if and only if (iff) it holds for a nominal (unperturbed) one,*

$$|\alpha_{1i,N}|^2 \rightarrow 0 \quad \textit{iff} \quad \lim_{N\to\infty} |\alpha^\circ_{1i,N}|^2 = 0 \quad (17)$$

If perturbations are non-Gaussian, (17) holds if $c_i = \mathbb{E}\{e^{j\Delta\Psi_{in}}\} \neq 0$. Otherwise, if $c_i = 0$, the FP always holds for the perturbed array, even if it does not hold for the nominal one. The following convergence holds in all considered cases:

$$|\alpha_{1i,N}|^2 \rightarrow \mathbb{E}\{|\alpha_{1i,N}|^2\} \quad (18)$$
$$= |c_i|^2 |\alpha^\circ_{1i,N}|^2 + (1 - |c_i|^2) N^{-1} \quad (19)$$
$$\rightarrow |c_i|^2 |\alpha^\circ_{1i,N}|^2 \quad (20)$$

Proof. First, we prove (18) in the MSE sense. To simplify notations, let $\alpha_N = \alpha_{1i,N}$, $\alpha^\circ_N = \alpha^\circ_{1i,N}$. Using (13) with $z_N = |\alpha_N|^2$, $a_N = \mathbb{E}\{|\alpha_N|^2\}$, note that it is equivalent to

$$\mathrm{Var}\{|\alpha_N|^2\} = \mathbb{E}\{|\alpha_N|^4\} - (\mathbb{E}\{|\alpha_N|^2\})^2 \rightarrow 0 \quad (21)$$

as $N \rightarrow \infty$. Finding the variance in (21) is rather involved since the fourth moment analysis is complicated. Hence, we present an upper bound and prove that it tends to zero as $N \rightarrow \infty$. To this end, let us show that $\mathbb{E}\{|\alpha_N|^4\}$ can be "sandwiched" via $|\mathbb{E}\{\alpha_N\}|^4$.

Lemma 1. *$\mathbb{E}\{|\alpha_N|^4\}$ can be bounded as follows:*

$$|\mathbb{E}\{\alpha_N\}|^4 \leq \mathbb{E}\{|\alpha_N|^4\} \leq |\mathbb{E}\{\alpha_N\}|^4 + 12N^{-1} \quad (22)$$

Proof. First, we prove the upper bound. Using (9), one obtains

$$\mathbb{E}\{|\alpha_N|^4\} = \frac{1}{N^4} \sum_{n_1..n_4} \mathbb{E}\{e^{j(\Psi_{in_1} - \Psi_{in_2} + \Psi_{in_3} - \Psi_{in_4})}\} \quad (23)$$

where $1 \leq n_k \leq N$. Since $e^{j\Psi_{in_k}}$ are independent for different n_k, we divide the total set $S_t = \{\{n_1, n_2, n_3, n_4\}, 1 \leq n_k \leq N\}$ into the set of distinct indices $S_d = \{\{n_1, n_2, n_3, n_4\}, n_i \neq n_j, \forall i \neq j\}$ and its complementary set $S^c_d = S_t - S_d$. To simplify the derivations, define

$$\beta_{i,n_1..n_4} = \prod_{k=1}^{4} \mathbb{E}\{z_{in_k}\}, \ \beta'_{i,n_1..n_4} = \mathbb{E}\{\prod_{k=1}^{4} z_{in_k}\} \quad (24)$$

where $z_{in_k} = e^{j(-1)^{k+1}\Psi_{in_k}}$. Using (23),

$$\mathbb{E}\{|\alpha_N|^4\} = \frac{1}{N^4} \sum_{n_1..n_4} \beta'_{i,n_1..n_4} \quad (25)$$
$$= \frac{1}{N^4}\{\sum_{n_1..n_4} \beta_{i,n_1..n_4} + \sum_{S^c_d} (\beta'_{i,n_1..n_4} - \beta_{i,n_1..n_4})\} \quad (26)$$
$$\leq |\mathbb{E}\{\alpha_N\}|^4 + N^{-4} \sum_{S^c_d} |\beta'_{i,n_1..n_4} - \beta_{i,n_1..n_4}| \quad (27)$$
$$\leq |\mathbb{E}\{\alpha_N\}|^4 + 12N^{-1} \quad (28)$$

where (27) is due to the triangle inequality. □

Next, a lower bound on $E\{|\alpha_N|^2\}$ follows from Jensen's inequality since $|\cdot|^2$ is convex [15, p. 229]:

$$\mathbb{E}\{|\alpha_N|^2\} \geq |\mathbb{E}\{\alpha_N\}|^2 \quad (29)$$

Using the equality in (21) in combination with (22) and (29),

$$\mathrm{Var}\{|\alpha_N|^2\} \leq 12N^{-1} \rightarrow 0 \quad (30)$$

Finally, using (30) and (13) with $z_N = |\alpha_N|^2$, $a_N = \mathbb{E}\{|\alpha_N|^2\}$, one obtains $|\alpha_N|^2 \xrightarrow{mse} \mathbb{E}\{|\alpha_N|^2\}$, which also implies $|\alpha_N|^2 \xrightarrow{\Pr} \mathbb{E}\{|\alpha_N|^2\}$.

The almost sure convergence, $|\alpha_N|^2 \xrightarrow{a.s.} \mathbb{E}\{|\alpha_N|^2\}$, follows, after some manipulations, from an extension of [17, Theorem 1] to two-dimensional sequences with non-zero means.

This establishes (18) for all 3 convergence modes. Let us now evaluate $\mathbb{E}\{|\alpha_N|^2\}$ to establish (19):

$$\mathbb{E}\{|\alpha_N|^2\} = N^{-1} + |\mathbb{E}\{\alpha_N\}|^2 - N^{-2}\sum_n |\mathbb{E}\{e^{j\Psi_{in}}\}|^2 \quad (31)$$

since Ψ_{in} are independent for different n, and

$$\mathbb{E}\{e^{j\Psi_{in}}\} = c_i e^{j\Psi_{in}^0}, \quad c_i = \mathbb{E}\{e^{j\Delta\Psi_{in}}\} \quad (32)$$

where $|c_i| \leq 1$ and c_i is independent of n since the distribution of $\Delta\Psi_{in}$ is independent of n (due to i.i.d. perturbations). Next, using (32) and (9), one obtains:

$$\mathbb{E}\{\alpha_N\} = N^{-1}\sum_{n=1}^{N} \mathbb{E}\{e^{j\Psi_{in}}\} = c_i\alpha_N^\circ \quad (33)$$

where (33) follows from (12). Using (31)-(33), (19) follows:

$$\mathbb{E}\{|\alpha_N|^2\} = N^{-1} + |c_i|^2|\alpha_N^\circ|^2 - N^{-1}|c_i|^2 \quad (34)$$

This implies $\lim_{N\to\infty}\mathbb{E}\{|\alpha_N|^2\} = |c_i|^2\lim_{N\to\infty}|\alpha_N^\circ|^2$. Therefore, if $c_i \neq 0$,

$$\lim_{N\to\infty}\mathbb{E}\{|\alpha_N|^2\} = 0 \quad \text{iff} \quad \lim_{N\to\infty}|\alpha_N^\circ|^2 = 0 \quad (35)$$

and, since $|\alpha_N|^2 \to \mathbb{E}\{|\alpha_N|^2\}$ as proved above, (17) follows. For Gaussian perturbations, from (43), $c_i = \mathbb{E}\{e^{j\Delta\Psi_{in}}\} = e^{-\delta^2/2} \neq 0$. For non-Gaussian perturbations, $c_i = 0$ is possible, in which case $|\alpha_N|^2 \to \mathbb{E}\{|\alpha_N|^2\} \to 0$, even if $\lim_{N\to\infty}|\alpha_N^\circ|^2 \neq 0$, as follows from (34). This completes the proof. □

IV. The Distribution and a Bound for $|\alpha_N|^2$

Eq. (18) implies that, for large N, $\mathbb{E}\{|\alpha_N|^2\}$ can be used as an estimate of actual IUI $|\alpha_N|^2 = |\alpha_{1i,N}|^2$. However, as Fig. 2 below shows, random fluctuations of $|\alpha_N|^2$ do exists for finite N and, hence, should be taken into account for a more reliable design. A simple way to accomplish this is via the following upper bound, which holds with high probability for large N and sufficiently-large m,

$$|\alpha_N|^2 \lessapprox |\alpha_N^{up}|^2 = \mathbb{E}\{|\alpha_N|^2\} + m\sigma_{|\alpha_N|^2} \quad (36)$$

where $\sigma^2_{|\alpha_N|^2} = \text{Var}\{|\alpha_N|^2\}$ is the variance, $m = 1..3$ controls the outage probability (i.e. the probability that $|\alpha_N|^2$ exceeds the bound $|\alpha_N^{up}|^2$) and the design is based on $|\alpha_N^{up}|^2$. As Fig. 2 below shows, using $m = 0$ is not sufficient and $m = 1..3$ provides more reliable design, with larger m corresponding to smaller outage probability.

To estimate $\sigma_{|\alpha_N|^2}$ for large N, note, from the central limit theorem [15, p. 406], that the real α_{N1} and imaginary α_{N2} parts of $\alpha_N = \alpha_{N1} + j\alpha_{N2}$ are asymptotically Gaussian and hence can be approximated, for large N, as

$$\alpha_{Nk} \sim \mathcal{N}(\mathbb{E}\{\alpha_{Nk}\}, \sigma^2_{Nk}), \ k = 1,2 \quad (37)$$

where $\sigma^2_{Nk} = \text{Var}\{\alpha_{Nk}\}$ and, using (33), $\mathbb{E}\{\alpha_N\} = c_i\alpha_N^0$. After some manipulations,

$$\sigma^2_{Nk} = \frac{1}{2}\sigma^2_N - \frac{(-1)^k}{2N}\text{Re}\{(c_i' - c_i^2)\beta_N^o\} \approx \frac{1}{2}\sigma^2_N \quad (38)$$

$$\sigma^2_N = \text{Var}\{\alpha_N\} = N^{-1}(1 - |c_i|^2), \ c_i' = \mathbb{E}\{e^{j2\Delta\Psi_{in}}\} \quad (39)$$

where $\beta_N^o = N^{-1}\sum_{n=1}^N e^{j2\Psi_n^0}$ is the nominal IUI leakage factor at double the frequency. One can further show that α_{N1} and α_{N2} are asymptotically uncorrelated and hence independent so that, for large N, the IUI is distributed as

$$|\alpha_N|^2 = \alpha^2_{N1} + \alpha^2_{N2} \sim 0.5\sigma^2_N\chi^2_2(\lambda) \quad (40)$$

where $\chi^2_2(\lambda)$ is the non-central chi-squared random variable with 2 degrees of freedom and the noncentrality parameter $\lambda = 2\sigma_N^{-2}|\mathbb{E}\{\alpha_N\}|^2$. Using the variance of $\chi^2_2(\lambda)$ in [18, p. 447], $\text{Var}\{|\alpha_N|^2\}$ can be approximated as

$$\sigma^2_{|\alpha_N|^2} = \text{Var}\{|\alpha_N|^2\} \approx \sigma^4_N + 2|c_i|^2|\alpha_N^0|^2\sigma^2_N \quad (41)$$

which can be used in (36) to evaluate $|\alpha_N^{up}|^2$. The accuracy of these approximations is examined in the next section.

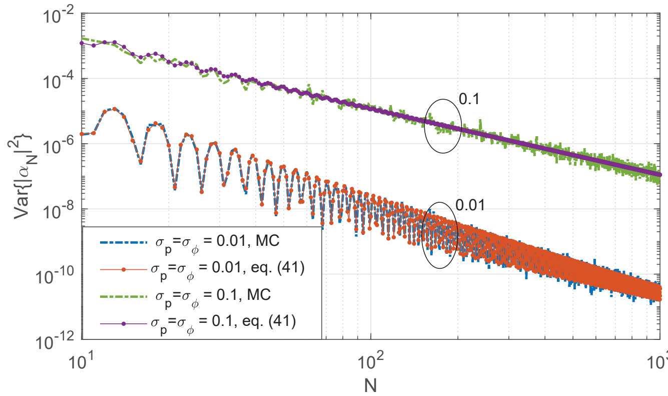

Fig. 1. Monte-Carlo (MC) simulated $\text{Var}\{|\alpha_N|^2\}$ and its approximation in (41) in the presence of zero-mean i.i.d. Gaussian errors of variances σ_p^2 and σ_ϕ^2, for the ULA with $d = 0.5$, the user angles-of-arrival are $\theta_1 = 0$, $\theta_i = \pi/8$ (with respect to the array broadside). The MC variance was evaluated over 100 trials generated independently for each N.

V. Examples and Discussion

To validate and illustrate the above results, consider a uniform linear array (ULA) with the nominal element spacing $d = 1/2$ measured in wavelengths. The perturbed array has location and phase errors, which are zero-mean i.i.d Gaussian of variances σ_p^2 and σ_ϕ^2. It follows that $\Delta\Psi_{in}$ is also zero mean Gaussian of variance

$$\delta^2 = 4\pi^2\sigma_p^2 + \sigma_\phi^2 \quad (42)$$

and therefore (see e.g. [12, p. 68])

$$c_i = \mathbb{E}\{e^{j\Delta\Psi_{in}}\} = e^{-\delta^2/2} \leq 1 \quad (43)$$

To validate the approximation in (41), Fig. 1 compares it to Monte-Carlo (MC) simulated $\text{Var}\{|\alpha_N|^2\}$. Note that the two agree well with each other over the whole range of N. The qualitatively-different behaviour of $\text{Var}\{|\alpha_N|^2\}$ for smaller and larger $\sigma_{p,\phi}$ can be explained using (41) as follows. In the large error regime $\sigma_{p,\phi} = 0.1$, σ_N^2 is larger and hence the first term σ_N^4 in (41) dominates, $\text{Var}\{|\alpha_N|^2\} \approx \sigma_N^4$, which decreases monotonically with N, as in (39). In the small error regime $\sigma_{p,\phi} = 0.01$, the 2nd term $2|c_i|^2|\alpha_N^0|^2\sigma_N^2$ in (41) dominates until about $N = 100$ and hence $\text{Var}\{|\alpha_N|^2\} \approx 2|c_i|^2|\alpha_N^0|^2\sigma_N^2$, which exhibits an oscillatory decrease with N due to $|\alpha_N^0|$.

Next, Fig. 2 illustrates the behaviour of the IUI factors as N increases. Clearly, the random IUI $|\alpha_N|^2$ (generated according to (9)-(11)), its mean $\mathbb{E}\{|\alpha_N|^2\}$ and nominal $|\alpha_N^\circ|^2$ values decrease with N, in agreement with Theorem 1. Note that $|\alpha_N|^2$ one exhibits statistical fluctuations due to random location and phase errors generated independently for each N, and $|\alpha_N^\circ|^2$ shows the fastest decrease. Also note significant difference between the random IUI $|\alpha_N|^2$ and its mean value $\mathbb{E}\{|\alpha_N|^2\}$, so that the latter can hardly serve as a reliable estimate of the former for finite N (even though both converge to 0 as $N \to \infty$). The more conservative upper bound in (36) accounts for random fluctuations and allows for more reliable design (with small outage probability); even $m = 1$ may not be sufficient, especially for $N > 100$, and $m = 3$ provides a more reliable design.

Fig. 3 illustrates the IUI factors for $\sigma_p = \sigma_\phi = 0.01$, i.e. in a small perturbation regime. Note that here, in a stark contrast to Fig. 2, all three, the random IUI, its mean and nominal values behave similarly until about $N = 100$, making the impact of random errors almost negligible. This can be explained via (18) and (19), whereby 1st term of (19) dominates in the small perturbation regime, for which $|c_i| \approx 1$ and therefore

$$(1 - |c_i|^2)N^{-1} \ll |c_i|^2|\alpha_N^\circ|^2 \tag{44}$$

so that, from (18), $|\alpha_N|^2 \approx \mathbb{E}\{|\alpha_N|^2\} \approx |\alpha_{1i,N}^\circ|^2$, i.e. the random and nominal IUI leakage factors are almost the same, making the impact of random perturbations negligible. Using (43) for Gaussian perturbations, (44) is equivalent to

$$\delta^2 \ll \ln(1 + N|\alpha_N^\circ|^2) \tag{45}$$

which quantifies the notion of small perturbation regime, where the impact of location and phase errors is negligible, i.e. all 3 IUI leakage factors are approximately the same. We caution the reader not to interpret (45) as that large errors are tolerable for larger N (and especially that arbitrarily-large errors are allowed as $N \to \infty$). The reason is that α_N° also depends on N and, in many cases, $|\alpha_N^\circ|^2 \sim N^{-2}$ so that the overall scaling of the upper bound in (45) is as $\ln(1 + N^{-1}) \sim N^{-1}$, i.e. just the opposite of what naive interpretation would suggest.

It follows from Theorem 1 that, if the FP holds for the nominal array, then $|\alpha_N|^2$, $\mathbb{E}\{|\alpha_N|^2\}$, $|\alpha_N^\circ|^2 \to 0$ as $N \to \infty$. Note, however, that while their convergence point is the same, the convergence speed is significantly different: while for the nominal array in many cases (e.g. an ULA with fixed element spacing, distinct AoAs and no grating lobes) $|\alpha_N^\circ|^2 \sim N^{-2}$, i.e. 20 dB per decade, for the perturbed one $|\alpha_N|^2$, $\mathbb{E}\{|\alpha_N|^2\} \sim N^{-1}$, i.e. 10 dB per decade, so that the impact of random errors, even if the FP holds, is to slow down the convergence from N^{-2} to N^{-1} and, hence, more antennas are needed to achieve the same low IUI leakage under random errors. As a further confirmation of this result, the $N = 100$ point in Fig. 2, where the IUI is about -20 dB under random errors, agrees well with the respective experimental results in [10, Fig. 11].

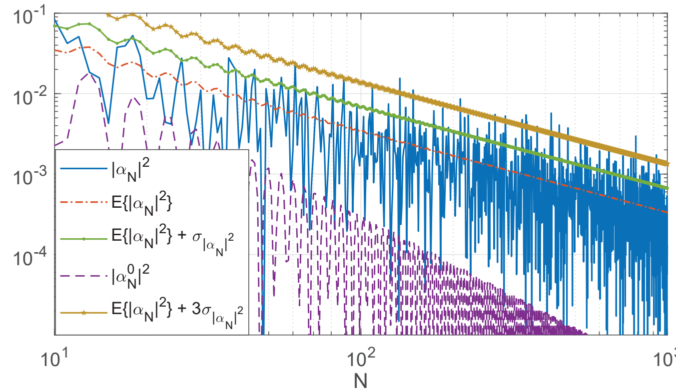

Fig. 2. IUI factor $|\alpha_N|^2$, its mean, estimated and nominal values for the N-element ULA with nominal element spacing $d = 0.5$, under zero-mean i.i.d. Gaussian perturbations with $\sigma_p = \sigma_\phi = 0.1$; $\theta_1 = 0$, $\theta_i = \pi/8$. While all 4 decrease with N, $|\alpha_N|^2$ fluctuates due to random errors (generated independently for each N) and $|\alpha_N^\circ|^2$ exhibits fastest decrease.

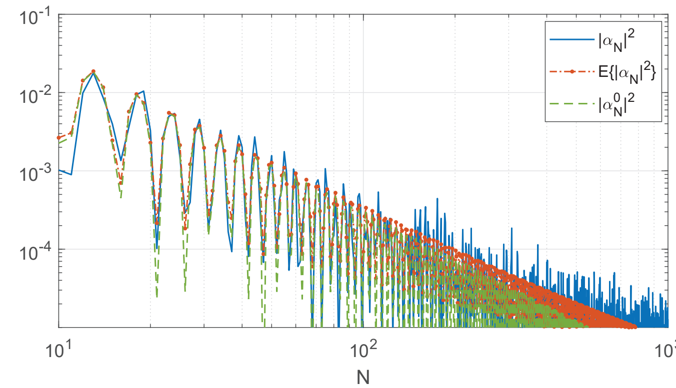

Fig. 3. The IUI factors as in Fig. 2 for $\sigma_p = \sigma_\phi = 0.01$ (small perturbation regime). Note that, unlike Fig. 2, all 3 behave similarly until about $N = 100$.

References

[1] T. L. Marzetta, Noncooperative Cellular Wireless With Unlimited Numbers of BS antennas, IEEE Trans. Wireless Comm., vol. 9, no. 11, pp. 3590–3600, Nov. 2010.

[2] H. Q. Ngo, E. G. Larsson, and T. L. Marzetta, Energy and Spectral Efficiency of Very Large Multiuser MIMO Systems, IEEE Trans. Comm., vol. 61, no. 4, pp. 1436–1449, Apr. 2013.

[3] L. Lu et al., An Overview of Massive MIMO: Benefits and Challenges, IEEE Journal Sel. Topics in Sig. Proc., v. 8, n.5, Oct. 2014.

[4] M. Shafi et al, 5G: A Tutorial Overview of Standards, Trials, Challenges, Deployment, and Practice, IEEE Journal Sel. Areas Comm., v. 35, N.6, pp. 1201-1221, June 2017.

[5] H. Tataria et al, 6G Wireless Systems: Vision, Requirements, Challenges, Insights, and Opportunities, Proc. IEEE, v. 109, no. 7, pp. 1166–1199, Jul. 2021.

[6] T. L. Marzetta et al., Fundamentals of Massive MIMO, Cambridge Univ. Press, 2016.

[7] H. Yang, T. L. Marzetta, Massive MIMO With Max-Min Power Control in Line-of-Sight Propagation Environment, IEEE Trans. Comm., vol. 65, no. 11, pp. 4685–4693, Nov. 2017.

[8] E. Anarakifirooz, S. Loyka, Favorable Propagation for Massive MIMO with Circular and Cylindrical Antenna Arrays, IEEE Wireless Comm. Letters, vol. 11, no. 3, pp. 458–462, Mar. 2022.

[9] X. Gao, O. Edfors, F. Rusek, and F. Tufvesson, Massive MIMO Performance Evaluation Based on Measured Propagation Data, IEEE Trans. Wireless Comm., vol. 14, no. 7, pp. 3899-3911, Jul. 2015.

[10] P. Harris et al., Performance Characterization of a Real-Time Massive MIMO System With LOS Mobile Channels, IEEE Journal Sel. Areas Comm., vol. 35, no. 6, pp. 1244–1253, June 2017.

[11] A. O. Martinez et al, An Experimental Study of Massive MIMO Properties in 5G scenarios, IEEE Trans. Antennas Propag., vol. 66, no. 12, pp. 7206–7215, Dec. 2018.

[12] H. L. Van Trees, Optimum Array Processing, Wiley, 2002.

[13] J. Li, P. Stoica, Robust Adaptive Beamforming. New York: Wiley, 2006.

[14] S. Loyka, C. D. Charalambous, Novel Matrix Singular Value Inequalities and Their Applications to Uncertain MIMO Channels, IEEE Trans. Info. Theory, vol. 61, no. 12, pp. 6623–6634, Dec. 2015.

[15] A. N. Shiryaev, Probability (3rd Ed.), New York, NY, USA: Springer-Verlag, 2015.

[16] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2, John Wiley & Sons, 1971.

[17] R. Lyons, Strong Laws of Large Numbers for Weakly Correlated Random Variables, Michigan Math. J., vol. 35, no. 3, pp. 353–359, 1988.

[18] N. L. Johnson et al, Continuous Univariate Distributions (2nd Ed.), New York: John Wiley & Sons, 1995.