# On The Capacity of Gaussian MIMO Channels Under Interference Constraints

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***Abstract*—Gaussian MIMO channel under total transmit and multiple interference power constraints (TPC and IPCs) is considered. A closed-form solution for its optimal transmit covariance matrix is obtained in the general case (up to dual variables). A number of more explicit closed-form solutions are obtained in some special cases, including full-rank and rank-1 (beamforming) solutions, which differ significantly from the well-known water-filling solutions (e.g. signaling on the channel eigenmodes is not optimal anymore and the capacity can be zero for non-zero transmit power). A whitening filter is shown to be an important part of optimal precoding under interference constraints. Capacity scaling with transmit power is studied: its qualitative behaviour is determined by a natural linear-algebraic structure induced by MIMO channels of multiple users. A simple rank condition is given to characterize the cases where spectrum sharing is possible. An interplay between the TPC and IPCs is investigated, including the transition from power-limited to interference-limited regimes. A number of unusual properties of an optimal covariance matrix under IPCs are pointed out and a bound on its rank is established. Partial null forming known in the adaptive antenna array literature is shown to be optimal from the information-theoretic perspective as well in some cases.**

## I. Introduction

Aggressive frequency re-use and hybrid (non-orthogonal) access schemes envisioned as key technologies in 5G systems [1] can potentially generate significant amount of inter-user interference and hence should be designed and managed carefully. In this respect, multi-antenna (MIMO) systems have significant potential due to their significant signal processing capabilities, including interference cancellation and precoding, which can also be done in an adaptive and distributed manner [2][3]. The capacity and optimal signalling for the Gaussian MIMO channel under the total power constraints (TPC) is well-known: the optimal (capacity-achieving) signaling is Gaussian and is on the eigenvectors of the channel with power allocation to the eigenmodes given by the water-filling (WF) [2]-[5]. Under per-antenna power constraints (PAC), in addition or instead of the TPC, Gaussian signalling is still optimal but not on the channel eigenvectors anymore so that the standard water-filling solution over the channel eigenmodes does not apply [6][7].

Much less is known if interference power constraints (IPC) are added, which limit the power of interference induced by a secondary transmitter to primary receivers (PR) in a spectrum-sharing system (e.g. cognitive radio). A game-theoretic approach to this problem was proposed in [8], where a fixed-point equation was formulated from which the optimal transmit covariance matrix can in principle be determined. Unfortunately, no closed-form solution is known for this equation and the considered settings require the channel to the primary receiver to be full-rank hence excluding the important cases where the number of Rx antennas is less than the number of Tx antennas (typical for massive MIMO downlink); the TPC is not included explicitly (rather, being "absorbed" into the IPC), hence eliminating the important case of inactive IPC and, consequently, no interplay between the TPC and the IPC can be studied.

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Cognitive radio MIMO systems under interference constraints have been also studied in [9]-[11], where a number of numerical optimization algorithms were developed but no closed-form solutions are known to the underlying optimization problems. Optimal signaling for the Gaussian MIMO channel under the TPC and the IPC has been also studied in [12]-[14] using the dual problem approach, and was extended to multi-user settings in [15]. However, the constraint matrices are required to be full-rank and no closed-form solution was obtained for optimal dual variables. Hence, various numerical algorithms or sub-optimal solutions were proposed. This limits insights significantly.

In this paper, we study the spectrum-sharing potential of Gaussian MIMO channels and concentrate on analysis rather than numerical algorithms. This provides deeper understanding of the problem and a number of insights unavailable from numerical algorithms alone. Specifically, we obtain novel closed-form solutions for an optimal transmit covariance matrix for the Gaussian MIMO channel under the TPC and multiple IPCs. All constraints are included explicitly and hence anyone is allowed to be inactive. This allows one to study the interplay between the power and interference constraints and, in particular, the transition from power-limited to interference limited regimes as the Tx power increases. As an added benefit, no limitation is placed on the rank of the constraint matrices, so that the number of antennas of the PR(s) can be any (including massive MIMO settings). In some cases, our approach leads to explicit closed-form solutions for optimal dual variables as well, including full-rank and rank-1 (beamforming) solutions and the conditions for their optimality. A whitening filter is shown to play a prominent role in optimal precoding under interference constraints. Partial null forming well-known in the antenna array literature [21] is shown to be optimal from the information-theoretic perspective as well, in certain cases. A simple rank condition is given to characterize the cases where spectrum sharing is possible for any interference power constraints. In general, the primary users have a major impact on the capacity at high SNR while being negligible at low SNR. The high-SNR behaviour of the capacity is qualitatively determined by the null spaces of PR channel matrices. The presented closed-form solutions of optimal signaling can be used directly in massive MIMO settings. Since numerical complexity of generic convex solvers can be prohibitively large for massive MIMO (in general, it scales as $m^6$ with the number $m$ of antennas), the above analytical solutions are a valuable low-complexity alternative.

It should be emphasized that, under the added IPCs, the unitary-invariance of the feasible set is lost and hence many known solutions and standard "tricks" (e.g. Hadamard inequality) of the analysis under the TPC alone cannot be used. This has profound impact on optimal signaling strategies as well as on analytical techniques to solve the underlying optimization problem. In particular, unlike the standard water-filling solution, (i) signaling on the channel eigenmodes is not optimal anymore (unless all IPCs are inactive or if their channel eigenmodes are the same as those of the main MIMO channel); (ii) the rank of an optimal Tx covariance matrix can exceed that of the channel; (iii) an optimal covariance matrix is not necessarily unique; (iv) the channel capacity can be zero for a non-zero Tx power and channel; (v) the channel capacity may stay bounded under unbounded growth of the Tx power (in which case the TPC is inactive). All these phenomena have major impact on the spectrum-sharing capabilities of MIMO channels. We demonstrate that the capacity scaling with the Tx power under multiple IPCs can be understood in terms of a natural linear-algebraic structure induced by the MIMO channels of different users.

*Notations*: bold capitals ($\boldsymbol{R}$) denote matrices while bold lower-case letters ($\boldsymbol{x}$) denote column vectors; $\boldsymbol{R}^+$ is the Hermitian conjugation of $\boldsymbol{R}$; $\boldsymbol{R} \geq 0$ means that $\boldsymbol{R}$ is positive semi-definite; $|\boldsymbol{R}|$, $tr(\boldsymbol{R})$, $r(\boldsymbol{R})$ denote determinant, trace and rank of $\boldsymbol{R}$, respectively; $\lambda_i(\boldsymbol{R})$ is $i$-th eigenvalue of $\boldsymbol{R}$; unless indicated otherwise, eigenvalues are in decreasing order, $\lambda_1 \geq \lambda_2 \geq ..$; $(x)_+ = \max[0, x]$ is the positive part of $x$; $\mathcal{R}(\boldsymbol{R})$ and $\mathcal{N}(\boldsymbol{R})$ denote the range and null space of $\boldsymbol{R}$ while $\boldsymbol{R}^\dagger$ is its Moore-Penrose pseudo-inverse; $\mathbb{E}\{\cdot\}$ is statistical expectation.

## II. Channel Model

Let us consider the standard discrete-time model of the Gaussian MIMO channel:

$$\boldsymbol{y}_1 = \boldsymbol{H}_1 \boldsymbol{x} + \boldsymbol{\xi}_1 \tag{1}$$

where $\boldsymbol{y}_1, \boldsymbol{x}, \boldsymbol{\xi}_1$ and $\boldsymbol{H}_1$ are the received and transmitted signals, noise and channel matrix. This is illustrated in Fig. 1. The noise is assumed to be complex Gaussian with zero mean and unit variance, so that the SNR equals to the signal power. A complex-valued channel model is assumed throughout the paper, with full channel state information available both at the transmitter and the receiver. Gaussian signaling is known to be optimal in this setting [2]-[5] so that finding the channel capacity $C$ amounts to finding an optimal transmit covariance matrix $\boldsymbol{R}$, which can be expressed as the following optimization problem (P1):

$$(P1): \ C = \max_{\boldsymbol{R} \in S_R} C(\boldsymbol{R}) \tag{2}$$

where $C(\boldsymbol{R}) = \log|\boldsymbol{I} + \boldsymbol{W}_1 \boldsymbol{R}|$, $\boldsymbol{W}_1 = \boldsymbol{H}_1^+ \boldsymbol{H}_1$ is the channel Gram matrix, $\boldsymbol{R}$ is the Tx covariance matrix and $S_R$ is the constraint (feasible) set. In the case of the total power constraint (TPC) only, it takes the form

$$S_R = S_{TPC} \triangleq \{\boldsymbol{R} : \boldsymbol{R} \geq 0, \mathrm{tr}(\boldsymbol{R}) \leq P_T\}, \tag{3}$$

where $P_T$ is the maximum total Tx power. The solution to this problem is well-known: optimal signaling is on the eigenmodes of $\boldsymbol{W}_1$, so that they are also the eigenmodes of optimal covariance $\boldsymbol{R}^* = \boldsymbol{R}_{WF}$, and the optimal power allocation is via the water-filling (WF). This solution can be compactly expressed as follows:

$$\boldsymbol{R}_{WF} \triangleq (\mu^{-1}\boldsymbol{I} - \boldsymbol{W}_1^{-1})_+ = \sum_{i:\lambda_{1i} > \mu} (\mu^{-1} - \lambda_{1i}^{-1}) \boldsymbol{u}_{1i} \boldsymbol{u}_{1i}^+$$

where $\mu \geq 0$ is the "water" level found from the total power constraint $tr(\boldsymbol{R}^*) = P_T$ (which is always active), $\lambda_{1i}, \ \boldsymbol{u}_{1i}$ are

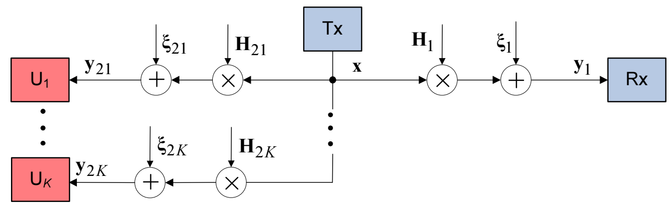


Fig. 1. A block diagram of multi-user Gaussian MIMO channel under interference constraints. $\boldsymbol{H}_1$ and $\boldsymbol{H}_{2k}$ are the channel matrices to the Rx and $k$-th user (PR) respectively. Interference constraints are to be satisfied for each user.

$i$-th eigenvalue and eigenvector of $\boldsymbol{W}_1$; $(\boldsymbol{R})_+$ denotes positive eigenmodes of Hermitian matrix $\boldsymbol{R}$: $(\boldsymbol{R})_+ = \sum_{i:\lambda_i > 0} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^+$, where $\lambda_i, \ \boldsymbol{u}_i$ are $i$-th eigenvalue and eigenvector of $\boldsymbol{R}$.

In a spectrum-sharing multi-user system (e.g. cognitive radio), there is a limit on how much interference the Tx can induce (via $\boldsymbol{x}$) to primary user $U_k$, see Fig. 1,

$$\mathbb{E}\{\boldsymbol{x}^+ \boldsymbol{H}_{2k}^+ \boldsymbol{H}_{2k} \boldsymbol{x}\} = tr(\boldsymbol{H}_{2k} \boldsymbol{R} \boldsymbol{H}_{2k}^+) \leq P_{Ik} \tag{4}$$

where $P_{Ik}$ is the maximum acceptable interference power and the left-hand side is the actual interference power at user $U_k$. In this setting, the feasible set becomes

$$S_R = \{\boldsymbol{R} \geq 0 : \ tr(\boldsymbol{R}) \leq P_T, \ tr(\boldsymbol{W}_{2k} \boldsymbol{R}) \leq P_{Ik} \ \forall k\} \tag{5}$$

where $\boldsymbol{W}_{2k} = \boldsymbol{H}_{2k}^+ \boldsymbol{H}_{2k}$ and $P_{Ik}$ are the channel Gram matrix of user $k$ and the respective interference constraint power, $k = 1..K$, where $K$ is the number of primary users.

The Gaussian signalling is still optimal in this setting and the capacity subject to the TPC and IPCs can still be expressed as in (2) but the optimal covariance is not $\boldsymbol{R}_{WF}$ anymore. In particular, the unitary-invariance of the feasible set $S_{TPC}$ under the TPC alone is lost due to the presence of the IPCs $tr(\boldsymbol{W}_{2k} \boldsymbol{R}) \leq P_{Ik}$ in $S_R$ so that well-known results and "tricks" (based on unitary invariance of the feasible set) cannot be used anymore. Since the "shape" of the feasible set $S_R$ affects significantly optimal $\boldsymbol{R}$, this results in a number of new properties of optimal signaling and of the capacity, as we show below.

One may also consider the total (rather than individual) interference power constraint so that

$$S_{RT} = \{\boldsymbol{R} : \boldsymbol{R} \geq 0, \ tr(\boldsymbol{R}) \leq P_T, \ \sum_k tr(\boldsymbol{W}_{2k} \boldsymbol{R}) \leq P_I\}$$

In this case, all the results of this paper will apply with $K = 1$, $P_{I1} = P_I$, and $\boldsymbol{W}_{21} \rightarrow \sum_k \boldsymbol{W}_{2k}$.

## III. Optimal Signalling Under the TPC and IPCs

To characterize fully the capacity, a closed-form solution for the optimal signaling problem (P1) in (2) under the joint constraints in (5) is given below in the general case, i.e. $\boldsymbol{W}_1, \boldsymbol{W}_{2k}$ are allowed to be singular and any of the constraints are allowed to be inactive. This extends the known results in [12]-[14] to the general case.

**Theorem 1.** *Consider the capacity of the Gaussian MIMO channel in* (2) *under the joint TPC and IPC in* (5). *The optimal Tx covariance matrix to achieve the capacity can be expressed as follows:*

$$\boldsymbol{R}^* = \boldsymbol{W}_\mu^\dagger (\boldsymbol{I} - \boldsymbol{W}_\mu \boldsymbol{W}_1^{-1} \boldsymbol{W}_\mu)_+ \boldsymbol{W}_\mu^\dagger \tag{6}$$

*where $\boldsymbol{W}_\mu = (\mu_1 \boldsymbol{I} + \sum_k \mu_{2k} \boldsymbol{W}_{2k})^{\frac{1}{2}}$; $\boldsymbol{W}_\mu^\dagger$ is the Moore-Penrose pseudo-inverse of $\boldsymbol{W}_\mu$; $\mu_1, \mu_{2k} \geq 0$ are Lagrange multipliers (dual variables) responsible for the TPC and IPCs, found from*

$$\mu_1(tr(\boldsymbol{R}^*) - P_T) = 0,\ \mu_{2k}(tr(\boldsymbol{W}_{2k}\boldsymbol{R}^*) - P_{Ik}) = 0 \qquad (7)$$

*subject to $tr(\boldsymbol{R}^*) \leq P_T$, $tr(\boldsymbol{W}_{2k}\boldsymbol{R}^*) \leq P_{Ik}$ $\forall k$. The respective capacity is*

$$C = \sum_{i:\lambda_{\mu i}>1} \log \lambda_{\mu i} \qquad (8)$$

*where $\lambda_{\mu i} = \lambda_i(\boldsymbol{W}_\mu^\dagger \boldsymbol{W}_1 \boldsymbol{W}_\mu^\dagger)$.*

*Proof.* See Appendix. □

Based on (6), one observes that $\boldsymbol{W}_\mu$ plays a role of a "whitening" filter, which disappears when all IPCs are inactive. When $\boldsymbol{W}_1$ is full-rank, i.e. $\boldsymbol{W}_1 > 0$, then $\boldsymbol{R}^*$ is unique, which is not necessarily the case in general - a remarkable difference to the TPC-only case, where $\boldsymbol{R}_{WF}$ is always unique. Dual variables $\mu_1, \mu_{2k}$ can be found numerically using the iterative bisection algorithm in [14]. In some special cases, closed-form solutions are possible - see Sections IV and V.

A number of known special cases follow from (6): If $K = 1$ and $\boldsymbol{W}_\mu$ is full-rank, then $\boldsymbol{W}_\mu^\dagger = \boldsymbol{W}_\mu^{-1}$ (see e.g. [17]) and $\boldsymbol{R}^*$ in (6) reduces to the respective solutions in [12]-[14]. If all IPCs are inactive, then $\mu_{2k} = 0$, $\boldsymbol{W}_\mu = \sqrt{\mu_1}\boldsymbol{I}$ and $\boldsymbol{R}^* = \boldsymbol{R}_{WF}$, as it should be.

### A. General properties

Next, we explore some general properties of the capacity related to its unbounded growth with $P_T$ and its being strictly positive. It turns out that those properties induce a natural linear-algebraic structure for the set of channels of all users.

It is well-known that, without the IPCs, $C(P_T)$ grows unbounded as $P_T$ increases, $C(P_T) \to \infty$ as $P_T \to \infty$ (assuming $\boldsymbol{W}_1 \neq 0$). This, however, is not necessarily the case under the added IPCs with all fixed $P_{Ik}$. The following proposition gives sufficient and necessary conditions when it is indeed the case.

**Proposition 1.** *Let $0 \leq P_{Ik} < \infty$ be fixed for all $k$. Then, the capacity grows unbounded as $P_T$ increases, i.e. $C(P_T) \to \infty$ as $P_T \to \infty$, if and only if*

$$\bigcap_k \mathcal{N}(\boldsymbol{W}_{2k}) \notin \mathcal{N}(\boldsymbol{W}_1) \qquad (9)$$

*or, equivalently,*

$$\mathcal{N}\big(\sum_k \boldsymbol{W}_{2k}\big) \notin \mathcal{N}(\boldsymbol{W}_1). \qquad (10)$$

*Proof.* See Appendix. □

The following observations are in order:

• Since the above conditions are both sufficient and necessary for the unbounded growth of the capacity, they give the exhaustive characterization of all the cases where such growth is possible. In practical terms, those cases represent the scenarios where any high spectral efficiency is achievable given enough power budget.

• The unbounded growth of the capacity with $P_T$ depends only on $\mathcal{N}(\sum_k \boldsymbol{W}_{2k})$ and $\mathcal{N}(\boldsymbol{W}_1)$, all other details being irrelevant.

• It can be seen that the condition $\mathcal{N}(\sum_k \boldsymbol{W}_{2k}) \notin \mathcal{N}(\boldsymbol{W}_1)$ holds if $r(\sum_k \boldsymbol{W}_{2k}) < r(\boldsymbol{W}_1)$, and hence the capacity grows unbounded with $P_T$ under the latter condition.

• On the other hand, if $\mathcal{N}(\sum_k \boldsymbol{W}_{2k}) \in \mathcal{N}(\boldsymbol{W}_1)$, then very high spectral efficiency cannot be achieved even with unlimited power budget, due to the dominance of the IPCs. In particular, if $\bigcap_k \mathcal{N}(\boldsymbol{W}_{2k}) = \emptyset$ or, equivalently, $\sum_k \boldsymbol{W}_{2k} > 0$, then (9) is impossible and the capacity stays bounded, even for infinite $P_T$ - the whole signaling space is dominated by IPCs in this case.

In the standard Gaussian MIMO channel without the IPCs, $C = 0$ if either $P_T = 0$ or/and $\boldsymbol{W}_1 = 0$, i.e. in a trivial way. On the other hand, in the same channel under the TPC and IPCs, the capacity can be zero in non-trivial ways, as the following proposition shows. In practical terms, this characterizes the cases where interference constraints of primary users rule out any positive rate of a given user and, hence, spectrum sharing is not possible. To this end, let $\mathcal{K}_0 = \{k : P_{Ik} = 0\}$, i.e. a set of all primary users requiring no interference, $P_{Ik} = 0$.

**Proposition 2.** *Consider the Gaussian MIMO channel under the TPC and IPCs and let $P_T > 0$, $\boldsymbol{W}_1 \neq 0$. Its capacity is zero if and only if $P_{Ik} = 0$ for some $k$ and*

$$\mathcal{N}\big(\sum_{k\in\mathcal{K}_0} \boldsymbol{W}_{2k}\big) \in \mathcal{N}(\boldsymbol{W}_1). \qquad (11)$$

*Proof.* See the full version of this paper [22]. □

Note that the condition $P_{Ik} = 0$ is equivalent to zero-forcing transmission with respect to user $U_k$, i.e. the capacity is zero only if ZF transmission is required for at least one user; otherwise, $C > 0$. The condition in (11) cannot be satisfied if $r(\boldsymbol{W}_1) > r(\sum_k \boldsymbol{W}_{2k})$ and hence $C > 0$ under the latter condition, which is also sufficient for unbounded growth of the capacity with $P_T$. This is summarized below.

**Corollary 1.** *If $r(\boldsymbol{W}_1) > r(\sum_k \boldsymbol{W}_{2k})$, then*

1. *$C \neq 0$ $\forall$ $P_{Ik} \geq 0$ and $P_T > 0$.*
2. *$C(P_T) \to \infty$ as $P_T \to \infty$ $\forall$ $P_{Ik} \geq 0$*

Thus, the condition $r(\boldsymbol{W}_1) > r(\sum_k \boldsymbol{W}_{2k})$ represents favorable propagation scenarios where spectrum sharing is possible for any $P_{Ik}$ and arbitrary large capacity can be attained given enough Tx power budget.

Unlike the standard WF where the TPC is always active, it can be inactive under the IPCs, which is ultimately due to the interplay of interference and power constraints. The following proposition explores this in some details. To this end, we call a constraint "redundant" if it can be omitted without affecting the capacity[1].

**Proposition 3.** *The TPC is redundant only if*

$$\mathcal{N}(\sum_k \boldsymbol{W}_{2k}) \in \mathcal{N}(\boldsymbol{W}_1) \qquad (12)$$

*and is active otherwise. In particular, it is active (for any $P_T$ and $P_{Ik}$) if $r(\boldsymbol{W}_1) > r(\sum_k \boldsymbol{W}_{2k})$, e.g. if $\boldsymbol{W}_1$ is full-rank and $\sum_k \boldsymbol{W}_{2k}$ is rank-deficient.*

*Proof.* See the full version of this paper [22]. □

[1]"inactive" implies "redundant" but the converse is not true: for example, inactive TPC means $tr\boldsymbol{R}^* < P_T$ and this implies $\mu_1 = 0$ (from complementary slackness) so that it is also redundant (can be omitted without affecting the capacity), but $\mu_1 = 0$ does not imply $tr\boldsymbol{R}^* < P_T$ since $tr\boldsymbol{R}^* = P_T$ is also possible in some cases.

## IV. Full-rank solutions

While Theorem 1 establishes a closed-form solution for optimal covariance $\boldsymbol{R}^*$ in the general case, it is expressed via dual variables $\mu_1, \mu_{2k}$ for which no closed-form solution is known in general so they have to be found numerically using (7). This limits insights significantly. In this section, we explore the cases when the optimal covariance $\boldsymbol{R}^*$ is of full rank and obtain respective closed-form solutions. To this end, we set $K = 1$, $\boldsymbol{W}_2 = \boldsymbol{W}_{21}$, $P_I = P_{I1}$, $\mu_2 = \mu_{21}$. First, we consider an interference-limited regime, where the TPC is redundant and hence the IPC is active.

**Proposition 4.** *Let $\boldsymbol{W}_1, \boldsymbol{W}_2 > 0$ and $P_I$ be bounded as follows:*

$$m\lambda_1(\boldsymbol{W}_2\boldsymbol{W}_1^{-1}) - tr(\boldsymbol{W}_2\boldsymbol{W}_1^{-1}) < P_I \le \frac{m}{tr(\boldsymbol{W}_2^{-1})}(P_T + tr(\boldsymbol{W}_1^{-1})) - tr(\boldsymbol{W}_2\boldsymbol{W}_1^{-1}) \tag{13}$$

*then $\mu_1 = 0$, i.e. the TPC is redundant, $\boldsymbol{R}^*$ is of full-rank and is given by:*

$$\boldsymbol{R}^* = \mu_2^{-1}\boldsymbol{W}_2^{-1} - \boldsymbol{W}_1^{-1} \tag{14}$$

*where $\mu_2^{-1} = m^{-1}(P_I + tr(\boldsymbol{W}_2\boldsymbol{W}_1^{-1}))$. The capacity can be expressed as*

$$C = m\log((P_I + tr(\boldsymbol{W}_2\boldsymbol{W}_1^{-1}))/m) + \log\frac{|\boldsymbol{W}_1|}{|\boldsymbol{W}_2|} \tag{15}$$

*Proof.* See the full version of this paper [22]. □

Next, we explore the case where $\boldsymbol{W}_2$ is of rank 1. This models the case when a primary user has a single-antenna receiver or when its channel is a keyhole channel, see e.g. [19][20].

**Proposition 5.** *Let $\boldsymbol{W}_1$ be of full rank and $\boldsymbol{W}_2$ be of rank-1, so that $\boldsymbol{W}_2 = \lambda_2\boldsymbol{u}_2\boldsymbol{u}_2^+$, where $\lambda_2 > 0$ and $\boldsymbol{u}_2$ are its active eigenvalue and eigenvector. If*

$$P_I \ge P_{I,th} = m^{-1}\lambda_2(P_T + tr(\boldsymbol{W}_1^{-1})) - \lambda_2\boldsymbol{u}_2^+\boldsymbol{W}_1^{-1}\boldsymbol{u}_2$$
$$P_T > m\lambda_1(\boldsymbol{W}_1^{-1}) - tr(\boldsymbol{W}_1^{-1}) \tag{16}$$

*then the IPC is redundant, the optimal covariance is of full rank and is given by the standard WF solution,*

$$\boldsymbol{R}^* = \boldsymbol{R}^*_{WF} = \mu_{WF}^{-1}\boldsymbol{I} - \boldsymbol{W}_1^{-1} \tag{17}$$

*where $\mu_{WF}^{-1} = m^{-1}(P_T + tr(\boldsymbol{W}_1^{-1}))$. If*

$$\lambda_2\lambda_1(\boldsymbol{W}_1^{-1}) - \lambda_2\boldsymbol{u}_2^+\boldsymbol{W}_1^{-1}\boldsymbol{u}_2 < P_I < P_{I,th}, \tag{18}$$
$$P_T > m\lambda_2^{-1}P_I + m\boldsymbol{u}_2^+\boldsymbol{W}_1^{-1}\boldsymbol{u}_2 - tr(\boldsymbol{W}_1^{-1}) \tag{19}$$

*then the IPC and TPC are active, the optimal covariance is of full rank and is given by*

$$\boldsymbol{R}^* = \mu_1^{-1}\boldsymbol{I} - \boldsymbol{W}_1^{-1} - \alpha\boldsymbol{u}_2\boldsymbol{u}_2^+ \tag{20}$$

*where $\alpha = \mu_1^{-1} - (\mu_1 + \lambda_2\mu_2)^{-1}$, and $\mu_1,\ \mu_2 > 0$ are*

$$\mu_1 = (P_T - \lambda_2^{-1}P_I - \boldsymbol{u}_2^+\boldsymbol{W}_1^{-1}\boldsymbol{u}_2 + tr(\boldsymbol{W}_1^{-1}))^{-1}(m-1)$$
$$\mu_2 = (P_I + \lambda_2\boldsymbol{u}_2^+\boldsymbol{W}_1^{-1}\boldsymbol{u}_2)^{-1} - \lambda_2^{-1}\mu_1 \tag{21}$$

*Proof.* See the full version of this paper [22]. □

Note that the 1st two terms in (20) represent the standard WF solution while the last term is a correction due to the IPC, which is reminiscent of a partial null forming in an adaptive antenna array, see e.g. [21, Sec. 6.3.1]. Hence, partial null forming is also optimal from the information-theoretic perspective.

## V. Rank-1 Solutions

In this section, we explore the case when $\boldsymbol{W}_1$ is rank-one. As we show below, beamforming is optimal in this case. A practical appeal of this is due to its low-complexity implementation. Furthermore, rank-one $\boldsymbol{W}_1$ is also motivated by single-antenna mobile units while the base station is equipped with multiple antennas, or when the MIMO propagation channel is of degenerate nature resulting in a keyhole effect, see e.g. [19][20].

We begin with the following result which bounds the rank of optimal covariance in any case.

**Proposition 6.** *If the TPC is active or/and $\boldsymbol{W}_2$ is full-rank, then the rank of the optimal covariance $\boldsymbol{R}^*$ of the problem (P1) in (2) under the constraints in (5) is bounded as follows:*

$$r(\boldsymbol{R}^*) \le r(\boldsymbol{W}_1) \tag{22}$$

*If the TPC is redundant and $\boldsymbol{W}_2$ is rank-deficient, then there exists an optimal covariance $\boldsymbol{R}^*$ (not necessarily unique) of (P1) under the constraints in (5) that also satisfies this inequality.*

*Proof.* See [22]. 1st part of this Proposition also holds for $K > 1$, with $\boldsymbol{W}_2 \to \boldsymbol{W}_\mu$. □

**Corollary 2.** *If $\boldsymbol{W}_2$ is of full-rank or/and if the TPC is active, then the optimal covariance $\boldsymbol{R}^*$ is of full-rank only if $\boldsymbol{W}_1$ is of full-rank (i.e. rank-deficient $\boldsymbol{W}_1$ ensures that $\boldsymbol{R}^*$ is also rank-deficient).*

**Corollary 3.** *If $r(\boldsymbol{W}_1) = 1$, then $r(\boldsymbol{R}^*) = 1$, i.e. beamforming is optimal.*

Note that this rank (beamforming) property mimics the respective property for the standard WF. However, while signalling on the (only) active eigenvector of $\boldsymbol{W}_1$ is optimal under the standard WF (no IPC), it is not so when the IPC is active, as the following result shows. To this end, let $\boldsymbol{W}_1 = \lambda_1\boldsymbol{u}_1\boldsymbol{u}_1^+$, i.e. it is rank-1 with $\lambda_1 > 0, \boldsymbol{u}_1$ be the (only) active eigenvalue and eigenvector; $\gamma_I = P_I/P_T$ be the "interference-to-signal" ratio, and

$$\gamma_1 = \frac{\boldsymbol{u}_1^+\boldsymbol{W}_2^\dagger\boldsymbol{u}_1}{\boldsymbol{u}_1^+(\boldsymbol{W}_2^\dagger)^2\boldsymbol{u}_1},\ \gamma_2 = \boldsymbol{u}_1^+\boldsymbol{W}_2\boldsymbol{u}_1 \tag{23}$$

where $\boldsymbol{W}_2^\dagger$ is Moore-Penrose pseudo-inverse of $\boldsymbol{W}_2$; $\boldsymbol{W}_2^\dagger = \boldsymbol{W}_2^{-1}$ if $\boldsymbol{W}_2$ is full-rank [17].

**Proposition 7.** *Let $\boldsymbol{W}_1$ be rank-1.*

*1. If $\gamma_I < \gamma_1$, then the TPC is redundant and the optimal covariance can be expressed as follows*

$$\boldsymbol{R}^* = P_I\frac{\boldsymbol{W}_2^\dagger\boldsymbol{u}_1\boldsymbol{u}_1^+\boldsymbol{W}_2^\dagger}{\boldsymbol{u}_1^+\boldsymbol{W}_2^\dagger\boldsymbol{u}_1} \tag{24}$$

*The capacity is*

$$C = \log(1 + \lambda_1\alpha P_T) \tag{25}$$

*where $\alpha = \gamma_I\boldsymbol{u}_1^+\boldsymbol{W}_2^\dagger\boldsymbol{u}_1 < 1$.*

*2. If $\gamma_I \ge \gamma_2$, then the IPC is redundant and the standard WF solution applies: $\boldsymbol{R}^* = P_T\boldsymbol{u}_1\boldsymbol{u}_1^+$. This condition is also necessary for the optimality of $P_T\boldsymbol{u}_1\boldsymbol{u}_1^+$ under the TPC and IPC when $\boldsymbol{W}_1$ is rank-1. The capacity is as in (25) with $\alpha = 1$.*

*3. If $\gamma_1 \le \gamma_I < \gamma_2$, then both constraints are active. The optimal covariance is*

$$\boldsymbol{R}^* = P_T\frac{\boldsymbol{W}_{2\mu}^{-1}\boldsymbol{u}_1\boldsymbol{u}_1^+\boldsymbol{W}_{2\mu}^{-1}}{\boldsymbol{u}_1^+\boldsymbol{W}_{2\mu}^{-2}\boldsymbol{u}_1} \tag{26}$$

*where $\boldsymbol{W}_{2\mu} = \boldsymbol{I} + \mu_2 \boldsymbol{W}_2$, and $\mu_2 > 0$ is found from the IPC: $tr(\boldsymbol{W}_2\boldsymbol{R}^*) = P_I$. The capacity is as in (25) with*

$$\alpha = (\boldsymbol{u}_1^+ \boldsymbol{W}_{2\mu}^{-1}\boldsymbol{u}_1)^2 |\boldsymbol{W}_{2\mu}^{-1}\boldsymbol{u}_1|^{-2} \leq 1 \tag{27}$$

*with equality if and only if $\boldsymbol{u}_1$ is an eigenvector of $\boldsymbol{W}_2$.*

*Proof.* See the full version of this paper [22]. □

Note that the optimal signalling in case 1 is along the direction of $\boldsymbol{W}_2^\dagger \boldsymbol{u}_1$ and not that of $\boldsymbol{u}_1$ (unless $\boldsymbol{u}_1$ is also an eigenvector of $\boldsymbol{W}_2$), as would be the case for the standard WF with redundant IPC. In fact, $\boldsymbol{W}_2^\dagger$ plays a role of a "whitening" filter here. Similar observation applies to case 3, with $\boldsymbol{W}_2$ replaced by $\boldsymbol{W}_{2\mu}$. $\alpha$ in Proposition 7 quantifies power loss due to enforcing the IPC; $\alpha = 1$ means no power loss.

## VI. Appendix

### A. Proof of Theorem 1

Since the problem is convex and Slater's condition holds, the KKT conditions are both sufficient and necessary for optimality [16]. They take the following form:

$$-(\boldsymbol{I} + \boldsymbol{W}_1\boldsymbol{R})^{-1}\boldsymbol{W}_1 - \boldsymbol{M} + \mu_1 \boldsymbol{I} + \sum_k \mu_{2k}\boldsymbol{W}_{2k} = 0 \tag{28}$$

$$\boldsymbol{M}\boldsymbol{R} = 0,\ \mu_1(tr(\boldsymbol{R}) - P_T) = 0,$$

$$\mu_{2k}(tr(\boldsymbol{W}_{2k}\boldsymbol{R}) - P_{Ik}) = 0, \tag{29}$$

$$\boldsymbol{M} \geq 0,\ \mu_1 \geq 0,\ \mu_{2k} \geq 0 \tag{30}$$

$$tr(\boldsymbol{R}) \leq P_T,\ tr(\boldsymbol{W}_{2k}\boldsymbol{R}) \leq P_{Ik},\ \boldsymbol{R} \geq 0 \tag{31}$$

where $\boldsymbol{M}$ is Lagrange multiplier responsible for the positive semi-definite constraint $\boldsymbol{R} \geq 0$. We consider first the case of full-rank $\boldsymbol{W}_\mu$ (i.e. either $\mu_1 > 0$ or/and $\sum_k \mu_{2k}\boldsymbol{W}_2 > 0$), so that $\boldsymbol{W}_\mu^\dagger = \boldsymbol{W}_\mu^{-1}$. Let us introduce new variables: $\tilde{\boldsymbol{R}} = \boldsymbol{W}_\mu \boldsymbol{R}\boldsymbol{W}_\mu$, $\tilde{\boldsymbol{W}}_1 = \boldsymbol{W}_\mu^{-1}\boldsymbol{W}_1\boldsymbol{W}_\mu^{-1}$, $\tilde{\boldsymbol{M}} = \boldsymbol{W}_\mu^{-1}\boldsymbol{M}\boldsymbol{W}_\mu^{-1}$. It follows that $\tilde{\boldsymbol{M}}\tilde{\boldsymbol{R}} = 0$ and (28) can be transformed to

$$(\boldsymbol{I} + \tilde{\boldsymbol{W}}_1\tilde{\boldsymbol{R}})^{-1}\tilde{\boldsymbol{W}}_1 + \tilde{\boldsymbol{M}} = \boldsymbol{I} \tag{32}$$

for which the solution is

$$\tilde{\boldsymbol{R}} = (\boldsymbol{I} - \tilde{\boldsymbol{M}})^{-1} - \tilde{\boldsymbol{W}}_1^{-1} = (\boldsymbol{I} - \tilde{\boldsymbol{W}}_1^{-1})_+ \tag{33}$$

(this can be established in the same way as for the standard WF). Transforming back to the original variables results in (6). (7) are complementary slackness conditions in (29); (8) follows, after some manipulations, by using $\boldsymbol{R}^*$ of (6) in $C(\boldsymbol{R})$.

The case of singular $\boldsymbol{W}_\mu$ is more involved. It implies $\mu_1 = 0$ so that $\boldsymbol{W}_\mu = (\sum_k \mu_{2k}\boldsymbol{W}_{2k})^{\frac{1}{2}}$. It follows from the KKT condition in (28) that, for the redundant TPC ($\mu_1 = 0$),

$$\boldsymbol{Q}_1(\boldsymbol{I} + \boldsymbol{Q}_1\boldsymbol{R}\boldsymbol{Q}_1)^{-1}\boldsymbol{Q}_1 + \boldsymbol{M} = \sum_k \mu_{2k}\boldsymbol{W}_{2k} \tag{34}$$

where $\boldsymbol{Q}_1 = \boldsymbol{W}_1^{1/2}$. Let $\boldsymbol{x} \in \mathcal{N}(\sum_k \mu_{2k}\boldsymbol{W}_{2k})$, i.e. $\sum_k \mu_{2k}\boldsymbol{W}_{2k}\boldsymbol{x} = 0$, then

$$\boldsymbol{x}^+\boldsymbol{Q}_1(\boldsymbol{I} + \boldsymbol{Q}_1\boldsymbol{R}\boldsymbol{Q}_1)^{-1}\boldsymbol{Q}_1\boldsymbol{x} + \boldsymbol{x}^+\boldsymbol{M}\boldsymbol{x} = 0 \tag{35}$$

so that $\boldsymbol{x}^+\boldsymbol{M}\boldsymbol{x} = 0$ and $\boldsymbol{Q}_1\boldsymbol{x} = 0$, since $\boldsymbol{M} \geq 0$ and $\boldsymbol{I} + \boldsymbol{Q}_1\boldsymbol{R}\boldsymbol{Q}_1 > 0$. Thus, $\mathcal{N}(\sum_k \mu_{2k}\boldsymbol{W}_{2k}) \in \mathcal{N}(\boldsymbol{Q}_1) = \mathcal{N}(\boldsymbol{W}_1)$ and $\mathcal{N}(\sum_k \mu_{2k}\boldsymbol{W}_{2k}) \in \mathcal{N}(\boldsymbol{M})$, i.e.

$$\mathcal{N}(\sum_k \mu_{2k}\boldsymbol{W}_{2k}) \in \mathcal{N}(\boldsymbol{W}_1) \cap \mathcal{N}(\boldsymbol{M}) \tag{36}$$

and this condition is also necessary for the TPC to be redundant. Further notice that

$$\mathcal{N}(\sum_k \mu_{2k}\boldsymbol{W}_{2k}) = \bigcap_{k\in\mathcal{K}_+} \mathcal{N}(\boldsymbol{W}_{2k}) = \mathcal{N}(\sum_{k\in\mathcal{K}_+} \boldsymbol{W}_{2k}) \tag{37}$$

where $\mathcal{K}_+ = \{k : \mu_{2k} > 0\}$ is the set of users with active IPCs. Let $\boldsymbol{W}_2 = \sum_k \mu_{2k}\boldsymbol{W}_{2k}$. Using (34), (36) and projecting all matrices on the active sub-space of $\boldsymbol{W}_2$, one can apply the solution in (6) with full-rank projected $\boldsymbol{W}_\mu$ (as established above) to the projected $\boldsymbol{R}$. Using this solution and transforming it back to the original basis, one obtains (6) after some manipulations - see [22] for further details.

### B. Proof of Proposition 1

To prove the "if" part, observe that $\bigcap_k \mathcal{N}(\boldsymbol{W}_{2k}) \notin \mathcal{N}(\boldsymbol{W}_1)$ implies $\exists \boldsymbol{u} : \boldsymbol{W}_{2k}\boldsymbol{u} = 0\ \forall k, \boldsymbol{W}_1\boldsymbol{u} \neq 0$. Now set $\boldsymbol{R} = P_T\boldsymbol{u}\boldsymbol{u}^+$, for which $tr(\boldsymbol{R}) = P_T, tr(\boldsymbol{W}_{2k}\boldsymbol{R}) = 0\ \forall k$, so it is feasible for any $P_T, P_{Ik}$. Furthermore,

$$C \geq C(\boldsymbol{R}) = \log(1 + P_T\boldsymbol{u}^+\boldsymbol{W}_1\boldsymbol{u}) \to \infty \tag{38}$$

as $P_T \to \infty$, since $\boldsymbol{u}^+\boldsymbol{W}_1\boldsymbol{u} > 0$.

Next, we will need the following technical result, which will also establish the last claim.

**Lemma 1.** *The following holds:*

$$\bigcap_k \mathcal{N}(\boldsymbol{W}_{2k}) = \mathcal{N}(\sum_k \boldsymbol{W}_{2k}) \tag{39}$$

*Proof.* See the full version of this paper [22]. □

To prove the "only if" part, let $\boldsymbol{W}_2 = \sum_k \boldsymbol{W}_{2k},\ P_I = \sum_k P_{Ik}$ and assume that $\mathcal{N}(\boldsymbol{W}_2) \in \mathcal{N}(\boldsymbol{W}_1)$. This implies that $\mathcal{R}(\boldsymbol{W}_1) \in \mathcal{R}(\boldsymbol{W}_2)$ (since $\mathcal{R}(\boldsymbol{W})$ is the complement of $\mathcal{N}(\boldsymbol{W})$ for Hermitian $\boldsymbol{W}$). Let

$$\boldsymbol{W}_k = \boldsymbol{U}_{k+}\boldsymbol{\Lambda}_k\boldsymbol{U}_{k+}^+,\ k = 1, 2 \tag{40}$$

where $\boldsymbol{U}_{k+}$ is a semi-unitary matrix of active eigenvectors of $\boldsymbol{W}_k$ and diagonal matrix $\boldsymbol{\Lambda}_k$ collects its strictly-positive eigenvalues. Notice that, from the IPC,

$$\begin{aligned} P_I &\geq tr(\boldsymbol{W}_2\boldsymbol{R}) = tr(\boldsymbol{\Lambda}_2\boldsymbol{U}_{2+}^+\boldsymbol{R}\boldsymbol{U}_{2+}) \\ &\geq \lambda_{r_2} tr(\boldsymbol{U}_{2+}^+\boldsymbol{R}\boldsymbol{U}_{2+}) \end{aligned} \tag{41}$$

where $\lambda_{r_2} > 0$ is the smallest positive eigenvalue of $\boldsymbol{W}_2$, so that

$$\lambda_1(\boldsymbol{U}_{2+}^+\boldsymbol{R}\boldsymbol{U}_{2+}) \leq P_I/\lambda_{r_2} < \infty \tag{42}$$

for any $P_T$. On the other hand, $\mathcal{R}(\boldsymbol{W}_1) \in \mathcal{R}(\boldsymbol{W}_2)$ implies $span\{\boldsymbol{U}_{1+}\} \in span\{\boldsymbol{U}_{2+}\}$ and hence

$$\lambda_1(\boldsymbol{U}_{1+}^+\boldsymbol{R}\boldsymbol{U}_{1+}) \leq \lambda_1(\boldsymbol{U}_{2+}^+\boldsymbol{R}\boldsymbol{U}_{2+}) \leq P_I/\lambda_{r_2} < \infty \tag{43}$$

so that

$$\begin{aligned} C(P_T) &= \log|\boldsymbol{I} + \boldsymbol{\Lambda}_1\boldsymbol{U}_{1+}^+\boldsymbol{R}^*\boldsymbol{U}_{1+}| \\ &= \sum_i \log(1 + \lambda_i(\boldsymbol{\Lambda}_1\boldsymbol{U}_{1+}^+\boldsymbol{R}^*\boldsymbol{U}_{1+})) \\ &\leq m\log(1 + \lambda_1(\boldsymbol{W}_1)\lambda_1(\boldsymbol{U}_{1+}^+\boldsymbol{R}^*\boldsymbol{U}_{1+})) \\ &\leq m\log(1 + \lambda_1(\boldsymbol{W}_1)P_I/\lambda_{r_2}) < \infty \end{aligned} \tag{44}$$

is bounded for any $P_T$, as required.

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