On the Capacity of Gaussian MIMO Channels Under the Joint Power Constraints

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***Abstract*—The capacity and optimal signaling over a fixed Gaussian MIMO channel are considered under the joint total and per-antenna power constraints. While the general case remains an open problem, a closed-form full-rank solution is obtained along with its sufficient and necessary conditions. The conditions for each constraint to be inactive are established. The high and low-SNR regimes are studied. Isotropic signaling is shown to be optimal in the former case while rank-1 signaling (beamforming) is not necessarily optimal in the latter case. Unusual properties of optimal covariance under the joint constraints are pointed out.**

***Index Terms*—MIMO, channel capacity, power constraint, optimal signalling.**

I. Introduction

MULTI-ANTENNA (MIMO) systems have been widely accepted by both academia and industry due to their high spectral efficiency. Presently, MIMO systems experience a re-surge of interest in the form of massive MIMO, which is considered a key technology for future 5G systems to meet ever-increasing traffic demand when a limited bandwidth is available [1]. The capacity of a fixed Gaussian MIMO channel and its optimal signaling strategy are well-known under the total transmit (Tx) power constaint (TPC): its is on the channel eigenmodes with power allocation given by the water-filling (WF) procedure [2], [3]. While the TPC is motivated by a limited power (energy) supply, individual per-antenna powers can also be limited when each antenna is equipped with its own amplifier (of limited power), in either collocated or distributed implementations, hence motivating per-antenna power constraint (PAC) for single-user as well as multi-user systems, as in [4]–[9]. While a number of iterative optimization algorithms have been proposed [4]–[6], closed-form solutions are known only in some special cases. The capacity and optimal signaling for a fixed Gaussian MISO channel under the PAC has been established in [7], which is significantly different from the standard WF solution and is equivalent to the equal-gain transmission (EGT) with phases adjusted to compensate for the channel phase shifts. This problems remains open in the general MIMO case while a numerical algorithm was proposed in [8] and a closed-form full-rank solution was obtained in [9].

The joint constraints, i.e., the TPC and the PAC simultaneously, are motivated by the scenario with limited overall power budget and where each antenna is equipped with its own power amplifier. The capacity of fixed Gaussian MISO channel under the joint TPC and PAC has been established in [10], where it was shown that the optimal signaling is a combination of EGT and maximum ratio transmission (MRT), with phase shifts adjusted to compensate channel-induced phase shifts. Following the remark in [9, Sec. II-B], the MISO result can be also adapted to any rank-1 MIMO channel. This result was further extended to fading MIMO channels, where it was shown that isotropic signaling is optimal if the fading distribution is right-unitary-invariant [10].

While an iterative algorithm to compute an optimal covariance was developed in [11] for the general MIMO case under the joint constraints, it provides limited insights (due to its iterative nature) and no closed-form solution is yet known. The key difficulty is the fact that, unlike the TPC only case, the feasible set of Tx covariance matrices is not isotropic anymore (due to the PAC) and hence the tools developed under the TPC (which exploit this symmetry) cannot be used anymore. New tools are needed.

This letter partially closes this gap by obtaining a closed-form full-rank solution for the optimal signaling (Tx covariance matrix) and respective capacity of a fixed full-rank Gaussian MIMO channel under the joint constraints when the constraint powers exceed certain thresholds, thus extending earlier analytical results in [9] and [10]. Sufficient and necessary conditions for optimal signaling to be of full rank are also established. Optimal signaling under the joint constraints is shown to have properties significantly different from those under the TPC only, see Section VI. It is the inter-play between the TPC and PAC that induces these unusual properties. The conditions when either TPC and PAC are inactive are given. The high and low-SNR regimes are studied. Isotropic signaling is shown to be optimal under the joint constraints in the former case while rank-1 signaling (beamforming) is not necessarily optimal in the latter case (in contrast to the standard WF signaling).

Notations: Bold lower-case letters denote column vectors while bold capital denote matrices; $\boldsymbol{R}^+$ is Hermitian conjugation of $\boldsymbol{R}$; r_{ii} denotes i-th diagonal entry of $\boldsymbol{R}$; $(\boldsymbol{R})_{ij}$ is ij-th entry of $\boldsymbol{R}$, $\lambda_i(\boldsymbol{R})$ is i-th eigenvalue of $\boldsymbol{R}$, unless indicated otherwise, eigenvalues are in decreasing order: $\lambda_1 \geq \lambda_2 \geq, \ldots,$; $\boldsymbol{R} \geq 0$ means that $\boldsymbol{R}$ is positive semi-definite; $|\boldsymbol{R}|$ is the determinant of $\boldsymbol{R}$, $\boldsymbol{I}$ is identity matrix of appropriate size.

II. Channel Model

Let us consider a discrete-time model of a fixed Gaussian MIMO channel:

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\xi} \tag{1}$$

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where $\boldsymbol{y}$, $\boldsymbol{x}$, $\boldsymbol{\xi}$ and $\boldsymbol{H}$ are the received and transmitted signals, noise and channel, respectively; m is the number of transmit antennas. The noise is Gaussian circularly-symmetric with zero mean and unit variance, so that power is also the SNR. The channel $\boldsymbol{H}$ is fixed and known to the transmitter and the receiver (Rx). Under the Tx power constraint(s), Gaussian signaling is known to be optimal in this setting [2], [3] so that finding the channel capacity C and optimal signaling amounts to finding an optimal Tx covariance matrix $\boldsymbol{R}$:

$$C = \max_{\boldsymbol{R}\in S_R} \ln |\boldsymbol{I} + \boldsymbol{W}\boldsymbol{R}| \tag{2}$$

where $\boldsymbol{W} = \boldsymbol{H}^+\boldsymbol{H}$, S_R is the constraint set. In the case of the TPC constraint only, it takes the form $S_R = \{\boldsymbol{R} : \boldsymbol{R} \geq 0,\ \mathrm{tr}\boldsymbol{R} \leq P_T\}$, where P_T is the maximum total Tx power, and the optimal covariance is well-known: the optimal signaling is on the channel eigenmodes with optimal power allocation via the water-filling, which can be compactly expressed as

$$\boldsymbol{R}^*_{WF} = (\lambda^{-1}\boldsymbol{I} - \boldsymbol{W}^{-1})_+ \tag{3}$$

where $(\boldsymbol{A})_+$ retains positive eigenmodes of Hermitian matrix,

$$(\boldsymbol{A})_+ = \sum_{i\,:\,\lambda_i(\boldsymbol{A})>0} \lambda_i(\boldsymbol{A})\boldsymbol{u}_i\boldsymbol{u}_i^+ \tag{4}$$

where $\boldsymbol{u}_i$ is i-th eigenvector of $\boldsymbol{A}$; $\lambda > 0$ is determined from the TPC $\mathrm{tr}\boldsymbol{R}^*_{WF} = P_T$.

Under the PA constraints, $S_R = \{\boldsymbol{R} : \boldsymbol{R} \geq 0, r_{ii} \leq P\}$, where r_{ii} is i-th diagonal entry of $\boldsymbol{R}$ (the Tx power of i-th antenna), P is the PA power constraint. No closed-form solution is known for the optimal covariance in the general case under this constraint, while such solutions are available in the MISO case [7] and in the MIMO case when the optimal covariance is of full-rank [9].

The joint power constraints, i.e., TPC and PAC, are motivated by practical designs where each antenna has its own amplifier (and hence PAC) while limited total power/energy supply motivates TPC. The optimal signaling and capacity have been obtained in closed form under the joint constraints for the MISO channel in [10], while the general MIMO case remains an open problem. The next section provides a closed-form full-rank solution for the MIMO case as well as sufficient and necessary conditions for this solution to hold and some related properties.

III. Optimal Signaling and Capacity

Following the standard arguments, see [2], [3], Gaussian signaling is still optimal under the joint constraints and the channel capacity C is as in (2), where the constraint set S_R is as follows:

$$S_R = \{\boldsymbol{R} : \boldsymbol{R} \geq 0, tr\boldsymbol{R} \leq P_T, r_{ii} \leq P\} \tag{5}$$

and P_T, P are the total and per-antenna constraint powers. Unfortunately, no closed-form solution is known for the optimal covariance in (2) under the constraints in (5) in the general case. The following Theorem partially closes this gap and gives a closed-form full-rank solution for optimal signaling in this setting. To this end, let $\boldsymbol{D}(\boldsymbol{W})$ be the diagonal matrix retaining only diagonal entries of $\boldsymbol{W}$ (with all off-diagonal entries set to zero), and $\bar{\boldsymbol{D}}(\boldsymbol{W}) = \boldsymbol{W} - \boldsymbol{D}(\boldsymbol{W})$ retaining off-diagonal entries only; $d_i = (\boldsymbol{W}^{-1})_{ii}$, where, without loss of generality, $d_1 \leq d_2 \leq, \ldots$, i.e., in decreasing order.

Theorem 1: Let the channel matrix in (1) be of full column rank, $\boldsymbol{W} = \boldsymbol{H}^+\boldsymbol{H} > 0$, and let the per-antenna and total transmit constraint powers be sufficiently high,

$$P > \lambda_m^{-1}(\boldsymbol{W}),\ P_T > m\lambda_m^{-1}(\boldsymbol{W}) - \mathrm{tr}\,\boldsymbol{W}^{-1} \tag{6}$$

Then, the (unique) optimal Tx covariance $\boldsymbol{R}^*$ in (2) under the TPC and PAC in (5) is of full-rank and is given by

$$\boldsymbol{R}^* = \min(P\boldsymbol{I}, \lambda^{-1}\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{W}^{-1})) - \bar{\boldsymbol{D}}(\boldsymbol{W}^{-1}) \tag{7}$$

$$= \lambda^{-1}\boldsymbol{I} - \boldsymbol{W}^{-1} - ((\lambda^{-1} - P)\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{W}^{-1}))_+ \tag{8}$$

where the operator min applies entry-wise; $\lambda = 0$ if $mP \leq P_T$, otherwise,

$$\lambda^{-1} = \frac{1}{m-k}\left(P_T + \sum_{i=k+1}^{m} d_i - kP\right), \tag{9}$$

k is the number of active PACs, determined as the largest integer satisfying

$$\alpha_k \geq mP - P_T \tag{10}$$

where $\alpha_k = \sum_{i=k}^{m}(d_i - d_k)$. The capacity is

$$C = \ln|\boldsymbol{W}| + \sum_{i=1}^{m} \ln\min(\lambda^{-1}, P + d_i) \tag{11}$$

Proof: See the Appendix. ■

Note that α_k is decreasing in k so that (10) can be efficiently solved (even for massive MIMO) by verifying the condition for k in increasing order and stopping at the largest k satisfying it. The number k of active PACs decreases with $mP - P_T$.

The expression in (8) has the following interpretation: its first part $\lambda^{-1}\boldsymbol{I} - \boldsymbol{W}^{-1}$ is the standard full-rank WF solution under the TPC only, and its 2nd part $(\lambda^{-1}\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{W}^{-1}) - P\boldsymbol{I})_+$ is a correction term accounting for the PAC.

It follows from (7) that per-antenna powers are as follows:

$$r_{ii} = \min(P, \lambda^{-1} - (\boldsymbol{W}^{-1})_{ii}) > 0 \tag{12}$$

which also has an insightful interpretation: these powers are the minimum of those under the PAC and TPC individually (1st and 2nd term in the min operator, respectively).

Next, we observe that the solution in Theorem 1 reduces to known solutions in some special cases.

Corollary 1: In Theorem 1, if the per-antenna constraint power P is sufficiently high,

$$P > m^{-1}(P_T + \alpha_1), \tag{13}$$

then all PACs are inactive and (8) reduces to the standard WF solution, $\boldsymbol{R}^* = \lambda^{-1}\boldsymbol{I} - \boldsymbol{W}^{-1}$, where $\lambda^{-1} = m^{-1}(P_T + \mathrm{tr}\,\boldsymbol{W}^{-1})$.

Corollary 2: In Theorem 1, if the TPC power P_T is sufficiently high, $P_T \geq m\ P$, then all PACs are active, the TPC is inactive and (7) reduces to the PAC-only full-rank solution in [9]: $\boldsymbol{R}^* = P\boldsymbol{I} - \bar{\boldsymbol{D}}(\boldsymbol{W}^{-1})$.

Corollary 3: In Theorem 1, *i*-th PAC is active if and only if

$$(\boldsymbol{W}^{-1})_{ii} < \lambda^{-1} - P \tag{14}$$

It should be pointed out that while (6) are sufficient for the optimal signaling to be of full-rank, they are not necessary, i.e., there are cases where the optimal signaling is of full-rank even when (6) does not hold. The following proposition gives necessary conditions for an optimal covariance to be of full-rank.

Proposition 1: Let $\boldsymbol{W} > 0$. The necessary conditions for optimal covariance $\boldsymbol{R}^*$ to be of full rank are as follows:

$$P > \lambda_1(\bar{\boldsymbol{D}}(\boldsymbol{W}^{-1})),\ \lambda < \lambda_m(\boldsymbol{W}) \tag{15}$$

where 1st condition is also sufficient if $mP \le P_T$ (inactive TPC), and λ is determined from (9).

Proof: Using (7), $\boldsymbol{R}^* \le P\boldsymbol{I} - \bar{\boldsymbol{D}}(\boldsymbol{W}^{-1})$, so that $\boldsymbol{R}^* > 0$ implies $P\boldsymbol{I} > \bar{\boldsymbol{D}}(\boldsymbol{W}^{-1})$ and hence 1st condition in (15). 2nd condition is obtained from $0 < \boldsymbol{R}^* \le \lambda^{-1}\boldsymbol{I} - \boldsymbol{W}^{-1}$. ■

Based on this, the following procedure can be used to establish whether optimal covariance is of full-rank in general:

1. If $P_T \ge mP$, then 1st condition in (15) is both sufficient and necessary for $\boldsymbol{R}^* > 0$ and (7) applies.

2. If $P_T < mP$, define $\boldsymbol{R}^*(\lambda)$ for a given $\lambda > 0$ from (7) and find λ from (9). If $\boldsymbol{R}(\lambda)^* > 0$, then it is a solution; otherwise, optimal covariance is rank-deficient.

This procedure gives an exhaustive characterization of all cases when $\boldsymbol{R}^*$ is of full rank for a full-rank channel (since it follows from KKT conditions which are necessary for optimality).

In the following, we characterize the conditions when some constraints are inactive for a full-rank channel (even when optimal covariance is not of full rank).

Proposition 2: Let $\boldsymbol{W} > 0$. If the TPC is inactive, then all PACs are active. Hence, (i) when at least one PAC is inactive, the TPC is active; (ii) the TPC is inactive if and only if $mP \le P_T$.

Proof: Follows from the stationarity condition in (24). ■

It should be noted that this Proposition does not hold if the channel is rank-deficient, as the example below demonstrates.

IV. High-SNR Regime

It is well-known that isotropic signaling is optimal at high SNR for the standard WF solution (under the TPC only) in a full-rank channel,

$$\boldsymbol{R}_{WF}^* \approx \frac{P_T}{m}\boldsymbol{I} \tag{16}$$

when $P_T \gg m\lambda_m^{-1}(\boldsymbol{W})$. In this section, we establish the optimality of isotropic signaling under the joint constraints. As a first step, the following proposition shows that isotropic signaling is optimal at high SNR under the PAC.

Proposition 3: Consider a full column-rank channel ($\boldsymbol{W} > 0$). Isotropic signaling is optimal in this channel under the PAC in the high-SNR regime, i.e., when $P \gg \lambda_m^{-1}(\boldsymbol{W})$,

$$\boldsymbol{R}_{PAC}^* \approx P\boldsymbol{I},\ C_{PAC} \approx \ln|\boldsymbol{W}| + m\ln P \tag{17}$$

Proof: First, observe that

$$C_{PAC} \ge C(P\boldsymbol{I}) \tag{18}$$

where $C(\boldsymbol{R}) = \ln|\boldsymbol{I} + \boldsymbol{W}\boldsymbol{R}|$, since $\boldsymbol{R} = P\boldsymbol{I}$ is feasible under the PAC. Next,

$$C_{PAC} \le C_{TPC}(mP) \tag{19}$$

where $C_{TPC}(mP)$ is the capacity under the TPC with the total power $P_T = mP$, since any feasible $\boldsymbol{R}$ under the PAC, $r_{ii} \le P$, is also feasible under the TPC with tr $\boldsymbol{R} \le mP$. Using (16), one obtains at high SNR $C_{TPC}(mP) \approx C(P\boldsymbol{I})$ and hence $C_{PAC} \approx C(P\boldsymbol{I})$ and (17) follow. ■

We are now in a position to establish the optimality of isotropic signaling under the joint (TPC + PAC) constraints at high SNR.

Proposition 4: Consider a full column-rank channel. Let $P^* = \min(P, P_T/m)$. Isotropic signaling is optimal in this channel under the joint constraints (TPC + PAC) in the high-SNR regime, i.e., when $P^* \gg \lambda_m^{-1}(\boldsymbol{W})$,

$$\boldsymbol{R}^* \approx P^*\boldsymbol{I},\ C \approx \ln|\boldsymbol{W}| + m\ln P^* \tag{20}$$

Proof: First, observe that $C \ge C(P^*\boldsymbol{I})$ since $\boldsymbol{R} = P^*\boldsymbol{I}$ is feasible under the joint constraints: $\mathrm{tr}\boldsymbol{R} \le P_T$ and $r_{ii} \le P$. Next,

$$C \le \min(C_{TPC}, C_{PAC}) \tag{21}$$

and, at high SNR, $C_{TPC} \approx C(P_T\boldsymbol{I}/m)$, $C_{PAC} \approx C(P\boldsymbol{I})$, and hence $C \approx C(P^*\boldsymbol{I})$, as desired. The inequality $P^* \gg \lambda_m^{-1}(\boldsymbol{W})$ comes from the approximation $\ln(1+x) \approx \ln x$, which holds if $x \gg 1$. ■

It is remarkable that, for any of the constraints considered here, isotropic signaling is optimal at high SNR. This simplifies the system design significantly as no feedback and no elaborate precoding are necessary for this signaling strategy. This also complements the respective result in [10] obtained for the right-unitary-invariant fading channel.

V. Low-SNR Regime

In this section, we consider the behaviour of optimal covariance in the low-SNR regime, namely, when

$$\min(mP, P_T) \ll \lambda_1^{-1}(\boldsymbol{W}) \tag{22}$$

It is well-known that, for the standard WF solution (under the TPC only), the optimal signaling is beamforming (rank-1) at low SNR, $\boldsymbol{R}_{WF}^* \approx P_T\boldsymbol{u}_1\boldsymbol{u}_1^+$, where $\boldsymbol{u}_1$ is the eigenvector of $\boldsymbol{W}$ corresponding to its largest eigenvalue. As the following example shows, this does not necessarily hold under the joint constraints.

Example: Let $P_T = 1.5 \cdot 10^{-2}$, $P = 10^{-2}$, and $\boldsymbol{W} = \mathrm{diag}\{2,1\}$. It is straightforward to see that the optimal covariance is $\boldsymbol{R}^* = 10^{-2} \cdot \mathrm{diag}\{1, 0.5\}$ in this case, i.e., full-rank and beamforming is not optimal, does not matter how low the SNR is. If, however, the per-antenna constraint power is increased to $P \ge 1.5 \cdot 10^{-2}$, all PACs become inactive and beamforming is optimal: $\boldsymbol{R}^* = 10^{-2} \cdot \mathrm{diag}\{1.5, 0\}$.

Hence, we conclude that it is the interplay between the TPC and the PAC that makes a significant difference at low SNR while having negligible impact at high SNR: while the optimal signaling under the TPC, the PAC and the joint constraints are all isotropic at high SNR, they are quite different at low SNR.

VI. Properties of Optimal Covariance

As it was shown in the previous sections, optimal signaling under the joint constraints can be significantly different from that under the TPC only. In the following, we point out additional significant differences.

1. The TPC can be inactive.

2. An optimal covariance is not necessarily unique (if $\boldsymbol{W}$ is rank-deficient).

3. An optimal covariance can be of full-rank even when the channel is not.

4. Optimal signaling is not on the eigenmodes of $\boldsymbol{W}$, unless it is diagonal or all PACs are inactive, and the capacity depends not only on its eigenvalues, but also on its eigenvectors.

These unusual properties should be contrasted with those under the TPC only, where (i) the TPC is always active (unless $C = 0$ - a trivial case not considered here), (ii) the optimal covariance is always unique, (iii) the optimal covariance is rank-deficient in a rank-deficient channel, and (iv) optimal signaling is on the channel eigenmodes and the capacity is independent of channel eigenvectors.

While Property 4 follows from Theorem 1, the following example illustrates Properties 1-3.

Example: Let $\boldsymbol{W} = \mathrm{diag}\{1,\, 0\}$, $P_T = 2$, $P = 1$. It is straightforward to see that

$$\boldsymbol{R}^* = \mathrm{diag}\{1, a\},\ 0 \le a \le 1 \tag{23}$$

so that (i) $\boldsymbol{R}^*$ is not unique, (ii) it is of full-rank when $a > 0$, even though the channel is not, and (iii) the TPC is inactive if $a < 1$ (so that minimizing a will minimize the total Tx power). Note however that if the channel is enhanced to a full-rank one, $\boldsymbol{W} = \{1,\, b\}$, $b > 0$, then $\boldsymbol{R}^* = \boldsymbol{I}$ and all unusual properties disappear.

Appendix
Proof of Theorem 1

Since the problem in (2) is convex and Slater's condition holds (as long as P, $P_T > 0$), its KKT conditions are both sufficient and necessary for optimality [12]. The KKT conditions for this problem are as follows:

$$-(\boldsymbol{I} + \boldsymbol{W}\boldsymbol{R})^{-1}\boldsymbol{W} - \boldsymbol{M} + \lambda\boldsymbol{I} + \boldsymbol{\Lambda} = 0 \tag{24}$$

$$\boldsymbol{M}\boldsymbol{R} = 0,\ \lambda(\mathrm{tr}\boldsymbol{R} - P_T) = 0,\ \lambda_i(r_{ii} - P) = 0 \tag{25}$$

$$\mathrm{tr}\boldsymbol{R} \le P_T,\ r_{ii} \le P,\ \boldsymbol{R} \ge 0 \tag{26}$$

$$\boldsymbol{M} \ge 0,\ \lambda \ge 0,\ \lambda_i \ge 0 \tag{27}$$

where $\lambda,\ \lambda_i$ are Lagrange multipliers (dual variables) responsible for the TPC and PAC, $\boldsymbol{M}$ is the (matrix) Lagrange multiplier responsible for $\boldsymbol{R} \ge 0$, $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_i\}$. The key difficulty in solving analytically these conditions is that they are a system of non-linear matrix equalities and inequalities, and the PACs make the feasible set S_R non-isotropic so that standard tools (e.g., Hadamard inequality) cannot be used. However, when $\boldsymbol{R}$ is of full rank, the stationarity condition simplifies to

$$(\boldsymbol{R} + \boldsymbol{W}^{-1})^{-1} = \lambda\boldsymbol{I} + \boldsymbol{\Lambda} \tag{28}$$

since $\boldsymbol{M} = 0$ (from $\boldsymbol{M}\boldsymbol{R} = 0$), so that

$$\boldsymbol{R} = (\lambda\boldsymbol{I} + \boldsymbol{\Lambda})^{-1} - \boldsymbol{W}^{-1} \tag{29}$$

where $\boldsymbol{\Lambda}$ is determined from the PACs

$$r_{ii} = (\lambda + \lambda_i)^{-1} - (\boldsymbol{W}^{-1})_{ii} \le P \tag{30}$$

and complementary slackness $\lambda_i(r_{ii} - P) = 0$ so that $\lambda_i > 0$ (active PAC) implies $r_{ii} = P$ and hence

$$\lambda_i = (P + (\boldsymbol{W}^{-1})_{ii})^{-1} - \lambda > 0 \tag{31}$$

Combining this with the case of inactive PAC $\lambda_i = 0$, one obtains $r_{ii} = \lambda^{-1} - (\boldsymbol{W}^{-1})_{ii} \le P$ and hence

$$\lambda_i = ((P + (\boldsymbol{W}^{-1})_{ii})^{-1} - \lambda)_+ \ge 0 \tag{32}$$

where $(x)_+ = \max(x, 0)$. It follows from (28) that off-diagonal parts of $\boldsymbol{R}$ and $\boldsymbol{W}^{-1}$ are the opposite of each other: $\bar{\boldsymbol{D}}(\boldsymbol{R}) = -\bar{\boldsymbol{D}}(\boldsymbol{W}^{-1})$ and, from (30)-(33), that

$$r_{ii} = \min(P, \lambda^{-1} - (\boldsymbol{W}^{-1})_{ii}) > 0 \tag{33}$$

from which (7) follows. Equation (8) is a straightforward manipulation of (7). Equation (9) follows from the TPC $\sum_i r_{ii} = P_T$ and (10) follows from $P \le \lambda^{-1} - (\boldsymbol{W}^{-1})_{ii}$ for all active PACs.

It remains to show that $\boldsymbol{R}^* > 0$. To this end, observe the following:

$$\begin{aligned}\boldsymbol{R}^* &= \min(P\boldsymbol{I}, \lambda^{-1}\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{W}^{-1})) - \bar{\boldsymbol{D}}(\boldsymbol{W}^{-1})\\ &> \min(\lambda_w^{-1}\boldsymbol{I}, \lambda_w^{-1}\boldsymbol{I} - \boldsymbol{D}(\boldsymbol{W}^{-1})) - \bar{\boldsymbol{D}}(\boldsymbol{W}^{-1})\\ &= \lambda_w^{-1}\boldsymbol{I} - \boldsymbol{W}^{-1} \ge 0\end{aligned} \tag{34}$$

where $\lambda_w = \lambda_m(\boldsymbol{W})$. The last inequality follows from $\lambda_w\boldsymbol{I} \le \boldsymbol{W}$, while 1st inequality follows from $P > \lambda_w^{-1}$ and $\lambda^{-1} > \lambda_w^{-1}$, where the latter inequality follows from (6) and (10) used in (9). This also implies that $r_{ii} > 0$ in (33). The uniqueness of $\boldsymbol{R}^*$ is due to the strict concavity of $\ln|\boldsymbol{I} + \boldsymbol{W}\boldsymbol{R}|$ when $\boldsymbol{W} > 0$.

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