Convexity of Error Rates in Digital Communications Under Non-Gaussian Noise

Sergey Loyka, Victoria Kostina, Francois Gagnon

***Abstract*—Convexity properties of error rates of a class of decoders, including the ML/min-distance one as a special case, are studied for arbitrary constellations. Earlier results obtained for the AWGN channel are extended to a wide class of (non-Gaussian) noise densities, including unimodal and spherically-invariant noise. Under these broad conditions, symbol error rates are shown to be convex functions of the SNR in the high-SNR regime with an explicitly-determined threshold, which depends only on the constellation dimensionality and minimum distance, thus enabling an application of the powerful tools of convex optimization to such digital communication systems in a rigorous way. It is the decreasing nature of the noise power density around the decision region boundaries that insures the convexity of symbol error rates in the general case. The known high/low SNR bounds of the convexity/concavity regions are tightened and no further improvement is shown to be possible in general.**

I. Introduction

Convexity properties play a well-known and important role in optimization problems [1], mainly due to two key reasons: (i) it is essentially the class of convex problems that are solvable numerically, and (ii) significant analytical insights are available for this class, which cannot be said about the general class of nonlinear problems.

In the world of digital communications, various types of error rates often serve as objective or constraint functions during optimization [2]-[3]. Therefore, their convexity properties are of considerable importance. While, in some simple scenarios, the convexity can be established by inspection or differentiation of corresponding closed-form error probability expressions, this approach is not feasible not only in the general case, but also in most cases of practical importance (e.g. modulation combined with coding etc.), since such expressions are either not known or prohibitively complex.

A general approach (i.e. not relying on particular closed-form probability of error expressions) to convexity analysis in binary detection problems has been developed in [4]. This approach has been later extended to arbitrary multidimensional constellations (which can also include coding) in [5]. In particular, it has been shown that the symbol error rate (SER) of the maximum-likelihood (ML) decoder operating in the AWGN channel is always convex in SNR in dimensions 1 and 2, and also in higher dimensions at high SNR and concave at low SNR (with explicitly specified boundaries of the high/low SNR regimes), for any modulation and coding. Bit error rate (BER) has also been shown to be convex in the high SNR regime [6], which fits the condition of vanishingly small probability of error in the channel coding theorem [7]. These results have been also extended to fading channels demonstrating that fading is never good in low dimensions.

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In the present paper, the earlier results in [5]-[7] are expanded in several directions, including an extension to a class of decoders and a wide class of noise densities, as well as tightening the earlier high/low SNR bounds of the convexity/concavity regions.

While the utility of the Gaussian noise model is well-known, there are a number of scenarios where it is not adequate, most notably an impulsive noise [8]-[12] with tails much heavier than Gaussian. To address this, an important and natural generalization of the Gaussian random process has been developed, namely, the spherically-invariant random process (SIRP). It has found a wide range of applications in communications, information-theoretic and signal processing areas [9]-[11]. While the marginal PDF of a SIRP may be significantly different from Gaussian, this class of processes shares a number of important theoretical properties with the Gaussian process: it is closed under linear transformations, it is the most general class of processes for which the optimal MMSE estimator is linear, and the optimal (ML) decoding is still the minimum distance one [9]-[11]. The present paper will extend this list to include the convexity properties of SER under a SIRP noise, which turn out to be similar to those in the AWGN channel. In addition, a general class of unimodal noise power densities will be considered and conditions on an arbitrary noise density will be formulated under which the SER is convex. In particular, the SER is convex in the SNR provided that the noise power density is decreasing around the decision region boundaries, regardless of its behavior elsewhere. It is convex at high SNR under a unimodal or a SIRP noise, and it is always convex (for any SNR) in low dimensions under SIRP noise. Similar results can also be obtained for convexity in signal amplitude and noise power (which are important for an equalizer design and a jammer optimization) and extended to fading channels and correlated noise.

The main contributions are as follows:

- New tighter high/low SNR bounds of the convexity/concavity regions are obtained and it is demonstrated that no further improvement is possible in the general case.
- While the earlier results in [5]-[7] were established for the ML (min-distance) decoders only, the same results are shown

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to apply to any decoder with center-convex decision regions, of which the min-distance one is a special case.

- While the earlier results were established for the AWGN channel only, the present paper deals with a wide class of noise densities (e.g. generic unimodal, SIRP etc.). In particular, the SER is shown to be convex at high SNR for this wider class as well; the SER turns out to be convex in low dimensions not only for the Gaussian, but also for an arbitrary SIRP noise. The constellation dimensionality and minimum distance appear as the main factors affecting the convexity properties.

II. System Model

The standard baseband discrete-time system model in an additive noise channel, which includes matched filtering and sampling, is

$$\mathbf{r} = \mathbf{s} + \boldsymbol{\xi} \tag{1}$$

where $\mathbf{s}$ and $\mathbf{r}$ are n-dimensional vectors representing transmitted and received symbols respectively, $\mathbf{s} \in \{\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_M\}$, a set of M constellation points, $\boldsymbol{\xi}$ is an additive white noise. Several general noise models will be considered, including the AWGN as a special case, for which $\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, \sigma_0^2 \mathbf{I})$, where σ_0^2 is the noise variance per dimension, and n is the constellation dimensionality; lower case bold letters denote vectors, bold capitals denote matrices, x_i denotes i-th component of $\mathbf{x}$, $|\mathbf{x}|$ denotes L_2 norm of $\mathbf{x}$, $|\mathbf{x}| = \sqrt{\mathbf{x}^T \mathbf{x}}$, where the superscript T denotes transpose, $\mathbf{x}_i$ denotes i-th vector. The average (over the constellation points) SNR is defined as $\gamma = 1/\sigma_0^2$, which implies the appropriate normalization, $\frac{1}{M}\sum_{i=1}^{M} |\mathbf{s}_i|^2 = 1$, unless indicated otherwise.

In addition to the maximum likelihood decoder (demodulator/detector), which is equivalent to the minimum distance one in the AWGN and some other channels [10][11],

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}_i} |\mathbf{r} - \mathbf{s}_i|,$$

a general class of decoders with center-convex decision regions (see Definition 1 and Fig. 1) will be considered, for which the min-distance one is a special case. The probability of symbol error P_{ei} given that $\mathbf{s} = \mathbf{s}_i$ was transmitted is

$$P_{ei} = \Pr[\hat{\mathbf{s}} \neq \mathbf{s}_i | \, \mathbf{s} = \mathbf{s}_i] = 1 - P_{ci} \tag{2}$$

where P_{ci} is the probability of correct decision, and the SER averaged over all constellation points is

$$P_e = \sum\nolimits_{i=1}^{M} P_{ei} \Pr[\mathbf{s} = \mathbf{s}_i] = 1 - P_c \tag{3}$$

where P_c is the overall probability of correct decision. Clearly, P_{ei} and P_{ci} possess the opposite convexity properties. P_{ei} can be expressed as

$$P_{ei} = 1 - \int_{\Omega_i} f_{\xi}(\mathbf{x}) d\mathbf{x} \tag{4}$$

where Ω_i is the decision region (Voronoi region), and $\mathbf{s}_i$ corresponds to $\mathbf{x} = 0$, i.e. the origin is shifted for convenience to the constellation point $\mathbf{s}_i$. For the min-distance decoder, Ω_i can be expressed as a convex polyhedron [1].

Note that the setup and error rate expressions we are using are general enough to apply to arbitrary multi-dimensional constellations, including coding (codewords are considered as points of an extended constellation). We now proceed to convexity properties of error rates in this general setting.

III. Convexity of Symbol Error Rates

Convexity properties of symbol error rates of the ML decoder in SNR and noise power have been established in [5] for arbitrary constellation/coding under ML decoding and AWGN noise and are summarized in Theorem 1 below for completeness and comparison purposes.

Theorem 1 (Theorems 1 and 2 in [5]): Consider the ML decoder operating in the AWGN channel. Its SER $P_e(\gamma)$ is a convex function of the SNR γ for any constellation/coding if $n \leq 2$,

$$d^2 P_e(\gamma)/d\gamma^2 = P_e(\gamma)'' \geq 0 \tag{5}$$

For $n > 2$, the following convexity properties hold:

* P_e is convex in the high SNR regime,

$$\gamma \geq (n + \sqrt{2n})/d_{\min}^2 \tag{6}$$

where $d_{\min} = \min_i\{d_{\min,i}\}$ is the minimum distance from a constellation point to the boundary of its decision region over the whole constellation, and $d_{\min,i}$ is the minimum distance from $\mathbf{s}_i$ to its decision region boundary,

* P_e is concave in the low SNR regime,

$$\gamma \leq (n - \sqrt{2n})/d_{\max}^2 \tag{7}$$

where $d_{\max} = \max_i\{d_{\max,i}\}$, and $d_{\max,i}$ is the maximum distance from $\mathbf{s}_i$ to its decision region boundary,

* there are an odd number of inflection points, $P_e(\gamma)'' = 0$, in the intermediate SNR regime,

$$(n - \sqrt{2n})/d_{\max}^2 \leq \gamma \leq (n + \sqrt{2n})/d_{\min}^2 \tag{8}$$

A. Convexity in SNR/Signal Power

Since the high/low SNR bounds in Theorem 1 are only sufficient for the corresponding property, a question arises whether they can be further improved. Theorem 2 provides such an improvement and demonstrates that no further improvement is possible.

Theorem 2: Consider the ML decoder operating in the AWGN channel. Its SER $P_e(\gamma)$ has the following convexity properties: it is convex in the high SNR regime,

$$\gamma \geq (n - 2)/d_{\min}^2 \tag{9}$$

it is concave in the low SNR regime,

$$\gamma \leq (n - 2)/d_{\max}^2 \tag{10}$$

and there are an odd number of inflection points in-between. The high/low SNR bounds cannot be further improved without further assumptions on the constellation geometry.

Proof: A key idea of the proof is to use the same technique as in [5] but in the spherical rather than Cartesian coordinates. The possibility of no further improvement is demonstrated via a constellation with all spherical decision

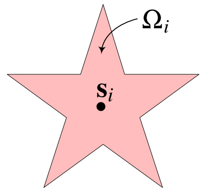

Fig. 1. Center-convex decision region Ω_i centered on the (given) $\mathbf{s}_i$.

regions, which achieve the equality in the bounds above. See [15] for details. ■

Note that the high/low SNR bounds in Theorem 2 are tighter than those in Theorem 1, since

$$n - \sqrt{2n} < n - 2 < n + \sqrt{2n} \text{ for } n > 2.$$

Convexity of the SER for $n \leq 2$ is also obvious from this Theorem. In the case of identical spherical decision regions, a more definite statement can be made.

Corollary 2.1: Consider the case of Theorem 2 when all decision regions are spheres of the same radius d. The following holds:

- The SER is strictly convex in γ in the high SNR regime:
$$P_e(\gamma)'' > 0 \text{ if } \gamma > (n-2)/d^2$$
- It is strictly concave in the low SNR regime:
$$P_i(\gamma)'' < 0 \text{ if } \gamma < (n-2)/d^2$$
- There is a single inflection point:
$$P_e(\gamma)'' = 0 \text{ iff } \gamma = (n-2)/d^2 \quad ■$$

Note that this result cannot be obtained from Theorem 1 directly, as the bounds there are not tight. It also follows from this Corollary that the high/low SNR bounds of Theorem 2 cannot be further improved in general (without further assumptions on the constellation geometry).

The results above are not limited to the AWGN channel but can also be extended to a wide class of noise densities and a class of decoders, as Theorem 3 below demonstrates. We will need the following definition generalizing the concept of a convex region.

Definition 1: A decision region is *center-convex* if a line segment connecting any of its points to a (given) center also belongs to the region (i.e. any point can be "seen" from the center).

Note that any convex region is automatically center-convex but the converse is not necessarily true, so that ML/min-distance decoders are a special case of a generic decoder with center-convex decision regions. As an example, Fig. 1 illustrates such a decision region, which is clearly not convex.

To generalize the results above to a wide class of noise densities, we transform the Cartesian noise density $f_\xi(\mathbf{x})$ into the spherical coordinates $(p, \boldsymbol{\theta})$, where $\boldsymbol{\theta} = \{\theta_1, .., \theta_{n-1}\}$ are the angles and p represents the normalized noise instant power $|\boldsymbol{\xi}|^2/\sigma_0^2$, and the new density is $f(p, \boldsymbol{\theta})$ (see [13][10] for more on spherical coordinates and corresponding transformations).

We are now in a position to generalize Theorem 2 to a wide class of noise densities and the class of center-convex decoders.

Theorem 3: Consider a decoder with center-convex decision regions operating in an additive noise channel of arbitrary density $f(p, \boldsymbol{\theta})$. The following holds:

$$P_e(\gamma)'' \geq 0 \text{ if } f'_p(p, \boldsymbol{\theta}) \leq 0 \; \forall \boldsymbol{\theta}, p \in [\gamma d^2_{\min}, \gamma d^2_{\max}], \quad (11)$$

where $f'_p(p, \boldsymbol{\theta}) = \partial f(p, \boldsymbol{\theta})/\partial p$. In particular, $P_e(\gamma)$ is convex in the interval $[\gamma_1, \gamma_2]$ if the noise density $f(p, \boldsymbol{\theta})$ is non-increasing in p in the interval $[\gamma_1 d^2_{\min}, \gamma_2 d^2_{\max}]$:

$$P_e(\gamma)'' \geq 0 \; \forall \gamma \in [\gamma_1, \gamma_2] \quad (12)$$
$$\text{if } f'_p(p, \boldsymbol{\theta}) \leq 0 \; \forall \boldsymbol{\theta}, p \in [\gamma_1 d^2_{\min}, \gamma_2 d^2_{\max}],$$

Proof: Follows along the same lines as that of Theorem 2 by performing the integration in (4) in spherical coordinates and setting $\gamma = 1/\sigma_0^2$ so that decision region boundaries are independent of the SNR [15]. ■

Note that it is the (non-increasing) behavior of the noise power density in the annulus $[\gamma_1 d^2_{\min}, \gamma_2 d^2_{\max}]$, i.e. around the boundaries of decision regions, that is responsible for the convexity of $P_e(\gamma)$; the behavior of the noise density elsewhere is irrelevant.

The inequalities in (11) and (12) can be reversed to obtain the corresponding concavity properties. The strict convexity properties can also be established by considering decoders with decision regions of non-zero measure in the corresponding SNR intervals. Convexity of individual SER P_{ei} can be obtained via the substitution $d_{\min(\max)} \rightarrow d_{\min,i(\max,i)}$. It is also straightforward to see that Theorem 2 is a special case of Theorem 3.

Let us now consider more special cases of Theorem 3.

Corollary 3.1: Consider a decoder with center-convex decision regions operating in an additive noise channel of a unimodal noise power density[1],

$$f'_p(p, \boldsymbol{\theta}) \begin{cases} > 0, & p < p^* \\ = 0, & p = p^* \\ < 0, & p > p^* \end{cases} \quad (13)$$

i.e. it has only one maximum at $p = p^*$; it is an increasing function on one side and decreasing on the other. Its SER is convex at high and concave at low SNR:

$$\begin{cases} P_e(\gamma)'' > 0, & \gamma > p^*/d^2_{\min} \\ P_e(\gamma)'' < 0, & \gamma < p^*/d^2_{\max} \end{cases} \quad (14)$$

■

Corollary 3.2: Consider the case of monotonically-decreasing (in p) noise power density, $f'_p(p, \boldsymbol{\theta}) < 0 \; \forall p, \boldsymbol{\theta}$. Then, the SER is always convex: $P_e(\gamma)'' > 0 \; \forall \gamma$. ■

Since the Gaussian noise power density is unimodal with $p^* = \max\{n-2, 0\}$, Corollary 3.1 applies to the AWGN channel as well, thereby generalizing Theorem 2 to decoders

[1] which is also quasi-concave; many popular probability density functions are unimodal [1].

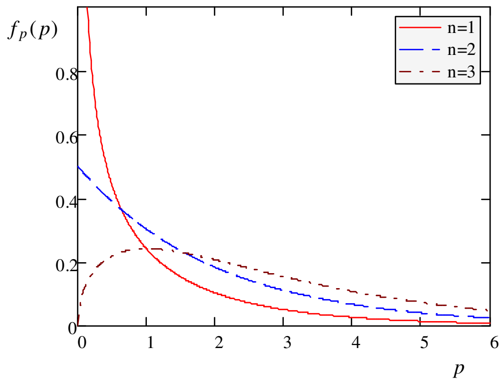

Fig. 2. The power density of Gaussian noise: while it is monotonically decreasing for $n = 1, 2$ and $\forall p$, it is unimodal for $n \geq 3$.

with center-convex decision regions. The AWGN for $n = 1, 2$ is also a special case of Corollary 3.2. These Corollaries allow one to answer the question "Why is the SER in the AWGN channel always convex for $n = 1, 2$ but not for $n \geq 3$?" - the reason is the monotonically decreasing (in p) nature of the noise power density $f(p, \boldsymbol{\theta})$ for any p in the former but not the latter case, see Fig. 2.

Other examples of unimodal densities include Laplacian, power exponential or Weibull distributions [11][14]. In fact, it was shown that Weibull distribution can be presented as a mixture of normal distributions, where the variance of a normal distribution is treated as a random variable with an α-stable distribution. This fits well into a typical model of interference in random wireless networks, where the interference distribution also follows an α-stable law [12]: each node transmits a Gaussian signal of a fixed transmit power; at the receiver, the noise power coming from each node is random and follows an α-stable law, so that the composite noise instant power follows the power exponential distribution.

B. Convexity of SER under SIRP noise

In this section, we consider an additive noise channel when the noise distribution follows that of a SIRP, which found a wide range of applications [9]-[11]. The characterization of the SIRP class is strikingly simple: any SIRP process is conditionally Gaussian, i.e. a Gaussian random process whose variance is a random variable independent of it. In the context of wireless communications, this structure represents such important phenomena as channel fading, random distance between transmitter and receiver, etc. Below, we establish the SER convexity properties under a SIRP noise, thus generalizing further the results of the previous section.

The following is one of the several equivalent definitions of a SIRP [9][10].

Definition 2: A random process $\{X(t), t \in R\}$ is a SIRP if a vector of any of its n samples $\mathbf{x} = \{X(t_1), X(t_2)..X(t_n)\}$ has the PDF of the following form:

$$f_{\mathbf{x}}(\mathbf{x}) = c_n h_n(\mathbf{x}^T \mathbf{C}_n^{-1} \mathbf{x}) \tag{15}$$

where $\mathbf{C}_n$ is the covariance matrix, $h_n(r)$ is a non-negative function of the scalar argument $r \geq 0$, and c_n is a normalizing constant. [2] ■

In fact, Definition 2 says that the PDF of SIRP samples depends only on the quadratic form $\mathbf{x}^T \mathbf{C}_n^{-1} \mathbf{x}$ rather than on each entry individually, so that any linear combinations of the entries of $\mathbf{x}$ having the same variance will also have the same PDF. Distributions of the functional form as in (15) are also known as elliptically-contoured distributions [13]. The characterization of SIRP is as follows (the SIRP representation theorem) [9][10].

Theorem 4: A random process is a SIRP iff any set of its samples has a PDF as in (15) with

$$h_n(r) = \int_0^{\infty} \sigma^{-n} \exp\left\{-\frac{r}{2\sigma^2}\right\} f(\sigma) d\sigma, \ \ 0 < r < \infty, \tag{16}$$

where $h_n(r)$ is defined by continuity at $r = 0$, and $f(\sigma)$ is any univariate PDF. ■

An equivalent representation is $X(t) = CY(t)$, where $Y(t)$ is the Gaussian random process of unit variance, and C is an independent random variable of PDF $f(\sigma)$, so that Theorem 4 basically says that any SIRP can be obtained by modulating the Gaussian random process by an independent random variable. A number of PDFs that satisfy Theorem 4 and corresponding $f(\sigma)$ can be found in [11] (which include Laplacian and power exponential densities above).

It was shown in [11] that the optimal decoder under the SIRP noise is still the minimum distance one (which follows from the fact that $h_n(r)$ in (16) is monotonically decreasing in r). Using this, we are now in a position to establish the SER convexity properties under SIRP noise with $\mathbf{C} = \mathbf{I}$.

Theorem 5: Consider an additive SIRP noise channel, where the noise density is as in (15) and (16) with $\mathbf{C} = \mathbf{I}$. Assume that $f(\sigma)$ in (16) has bounded support: $f(\sigma) = 0 \, \forall \sigma \notin [\sigma_1, \sigma_2]$. Then, the SER of any decoder with center-convex decision regions operating in this channel is convex at high and concave at low SNR as follows:

$$P_e(p_s)'' \geq 0 \text{ if } p_s \geq (n-2)\sigma_2^2/d_{\min}^2 \tag{17}$$

$$P_e(p_s)'' \leq 0 \text{ if } p_s \leq (n-2)\sigma_1^2/d_{\max}^2 \tag{18}$$

where p_s is the signal power, and $d_{\min(\max)}$ is the minimum (maximum) distance in the normalized constellation (corresponding to $p_s = 1$).

Proof: Follows along the same steps as that of Theorem 3 using the representation in (16) [15]. ■

Note that the high/low SNR bounds are independent of a particular form of $f(\sigma)$, but depend only on the corresponding boundaries of its support set. A particular utility of this Theorem is due to the fact that closed-form expressions of $P_e(p_s)$ are not available in most cases so its convexity cannot be evaluated directly. The following Corollary is immediate.

Corollary 5.1: Consider a decoder with center-convex decision regions operating in the SIRP noise channel as in

[2] An equivalent definition in terms of the characteristic function is also possible. Note also that not any $h_n(r)$ will do the job, but only those satisfying the Kolmogorov consistency condition [9][10].

Theorem 5 without the bounded support assumption. Its SER $P_e(p_s)$ is always convex when $n \leq 2$: $P_e(p_s)'' \geq 0 \; \forall p_s$. ■

Thus, the SER is convex in low dimensions for all the noise densities in Table I in [11] (i.e. contaminated normal, generalized Laplace, Cauchy and Gaussian), which extends the corresponding result in Theorem 1 to a generic SIRP noise.

C. Convexity in Signal Amplitude

Convexity of the SER as a function of signal amplitude $A = \sqrt{\gamma}$, $P_e(A)$, is also important for some optimization problems (e.g. an equalizer design). For the ML decoder operating in the AWGN channel those properties have been established in [5]. The next Theorem provides tighter high/low SNR bounds than those in [5], which cannot be further improved in general, and also extends the result to any decoder with center-convex decision regions. Due to the page limit, we skip the proofs and refer the reader to [15].

Theorem 6: Consider a decoder with center-convex decision regions operating in the AWGN channel. Its SER $P_{ei}(A)$ as a function of signal amplitude A has the following convexity properties for any n:

- The SER is convex in A in the large SNR regime:
$$P_{ei}(A)'' \geq 0 \text{ if } A \geq \sqrt{n-1}/d_{\min,i}$$
- It is concave in the small SNR regime
$$P_{ei}(A)'' \leq 0 \text{ if } A \leq \sqrt{n-1}/d_{\max,i}$$
- There are an odd number of inflection points in-between.
- The bounds cannot be further tightened in general (without further assumptions on the constellation geometry).

The convexity of $P_{ei}(A)$ for $n = 1$ and any A follows automatically from this Theorem, in addition to the following.

Corollary 6.1: Consider the case of Theorem 6 when all decision regions are the spheres of same radius d. The following holds:

- The SER is strictly convex in A in the large SNR regime:
$$P_e(A)'' > 0 \text{ if } A > \sqrt{n-1}/d$$
- It is strictly concave in the small SNR regime:
$$P_e(A)'' < 0 \text{ if } A < \sqrt{n-1}/d$$
- There is a single inflection point:
$$P_e(A)'' = 0 \text{ if } A = \sqrt{n-1}/d \quad \blacksquare$$

Theorem 6 can also be extended to a wide class of noise densities following the same approach as in Theorem 3.

Theorem 7: Consider a decoder with center-convex decision regions operating in an additive noise channel of arbitrary density $f(r, \boldsymbol{\theta})$, where r represents the normalized noise amplitude $|\boldsymbol{\xi}|/\sigma_0$. The SER $P_e(A)$ is convex in A in the interval $[A_1, A_2]$ if the noise density $f(r, \boldsymbol{\theta})$ is non-increasing in r in the interval $[A_1 d_{\min}, A_2 d_{\max}]$:

$$P_e(A)'' \geq 0 \; \forall A \in [A_1, A_2] \qquad (19)$$
$$\text{if } f_r'(r, \boldsymbol{\theta}) \leq 0 \; \forall \boldsymbol{\theta}, r \in [A_1 d_{\min}, A_2 d_{\max}] \quad \blacksquare$$

For the case of a SIRP noise as in Theorem 5, one obtains the following.

Theorem 8: Consider an additive SIRP noise channel with the density as in (15), (16) and $\mathbf{C} = \mathbf{I}$. Assume that $f(\sigma)$ has bounded support: $f(\sigma) = 0 \; \forall \sigma \notin [\sigma_1, \sigma_2]$. Then, the SER of any decoder with center-convex decision regions operating in this channel is convex at high SNR and concave at low SNR as a function of signal amplitude A:

$$P_e(A)'' \;\geq\; 0 \text{ if } A \geq \sigma_2 \sqrt{n-1}/d_{\min} \qquad (20)$$
$$P_e(A)'' \;\leq\; 0 \text{ if } A \leq \sigma_1 \sqrt{n-1}/d_{\max} \qquad (21)$$

where $d_{\min(\max)}$ is the minimum (maximum) distance of the normalized constellation (i.e. the one that corresponds to $A = 1$). ■

The following is immediate.

Corollary 8.1: Consider the scenario in Theorem 8 for $n = 1$. The SER is always convex in A: $P_e(A)'' \geq 0 \; \forall A$. ■

D. Extension to Correlated Noise and Fading

Finally, we note that all the results can also be extended to correlated/non-i.i.d. noise via the sufficient statistics approach and a whitening filter, and to fading channels using the same approach as in [5].

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