Bit Error Rate is Convex at High SNR

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MOTIVATION

- **Importance of convexity**[1]
  - o strong analytical structure (convex problem ~ linear problem)
  - o optimization problems: global solution
  - o powerful numerical/analytical techniques
  - o significant insight (even if no solution)
- **BER/SER is in the core of digital communications**
  - o important performance objective
  - o subject to analysis/optimization/design
- **Various applications/insights/generalizations**

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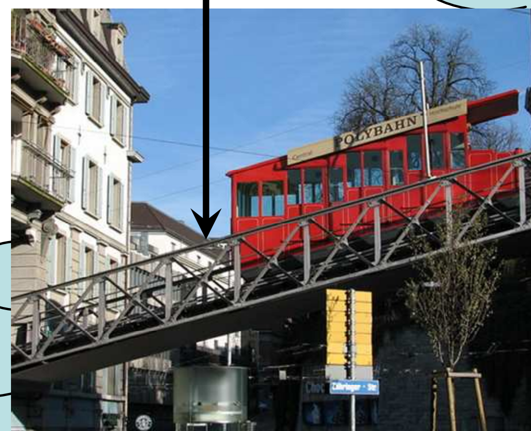
[1] see e.g. S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.

Digital Communications

bridge



Convex Analysis

- strong analytical structure
- analytical/numerical techniques
- algorithms
- global solutions
- convergence

SYSTEM MODEL: AWGN CHANNEL

Baseband, discrete-time AWGN channel

$$\mathbf{r} = \mathbf{s} + \boldsymbol{\xi}$$

$\mathbf{s}$, $\mathbf{r}$ are the Tx and Rx symbols,

$\mathbf{s} \in \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M\}$, a set of M symbols (constellation/codebook)

$\boldsymbol{\xi} \sim \mathcal{CN}(\mathbf{0}, \sigma_0^2 \mathbf{I})$ is AWGN,

$$p_{\xi}(\mathbf{x}) = \left(\frac{1}{2\pi\sigma_0^2} \right)^{\frac{n}{2}} \mathrm{e}^{-\frac{|\mathbf{x}|^2}{2\sigma_0^2}}$$

Fading channel can be considered as well.

MAXIMUM LIKELIHOOD (ML) DETECTOR

The maximum likelihood (ML) = minimum distance detector,

$$\hat{\mathbf{s}} = \arg\min_{\mathbf{s}} |\mathbf{y} - \mathbf{s}|$$

Symbol error rate (SER_i / SER),

$$P_{ei} = \mathsf{SER}_i = \Pr\left[\hat{\mathbf{s}} \neq \mathbf{s}_i \middle| \mathbf{s} = \mathbf{s}_i\right], \quad P_e = \mathsf{SER} = \sum_{i=1}^{M} P_{ei} \Pr\left[\mathbf{s} = \mathbf{s}_i\right]$$

Bit error rate (BER):

$$\mathsf{BER} = \sum_{i=1}^{M} \sum_{j \neq i} \frac{h_{ij}}{\log_2 M} \Pr\left\{\mathbf{s} = \mathbf{s}_i\right\} \Pr\left\{\mathbf{s}_i \to \mathbf{s}_j\right\}$$

PRIOR RESULTS[2]: CONVEXITY OF SER

Theorem 1:

- $n \le 2$: SER is *convex in low dimensions,* (any constellation/ coding), $d^2\mathrm{SER} / d\mathrm{SNR}^2 > 0$,
- $n > 2$:
  - SER is *convex at high SNR*, $\mathrm{SNR} \ge \left(n + \sqrt{2n}\right) / d_{\min}^2$
  - *concave at low*, $\mathrm{SNR} \le \left(n - \sqrt{2n}\right) / d_{\max}^2$
  - odd number of inflection points in-between.

[2] S. Loyka, V. Kostina, F. Gagnon, Error Rates of the Maximum-Likelihood Detector for Arbitrary Constellations: Convex/Concave Behavior and Applications, IEEE Trans. Info. Theory, 2009, accepted.

RECENT IMPROVEMENTS

Theorem 1a: High/low SNR bounds can be strengthened as

High SNR: $\text{SNR} \geq (n-2)/d_{\min}^2$

Low SNR: $\text{SNR} \leq (n-2)/d_{\max}^2$

and no further improvement is possible.

For spherical decision regions: sufficient *and* necessary.

Extensions:

- not only ML, any detector such that

$$\text{Sphere}(R = d_{\min}) \in \Omega_i \text{ (simply-bounded)}$$

- any unimodal noise power density (e.g. Laplace)

(BEAUTIFUL) BOUNDS ON SER DERIVATIVES IN SNR

$$-\frac{c_n}{\gamma} \le P'_{e|\gamma} \le 0, \quad , \quad c_1 = \frac{1}{\sqrt{2\pi e}}, \quad c_2 = \frac{1}{e}$$

$$\frac{\beta_l}{\gamma^2} \le P''_{e|\gamma} \le \frac{\beta_u}{\gamma^2}, \; n = 2: \; 0 \le P''_{e|\gamma} \le \left(\frac{2}{e\gamma} \right)^2$$

c_n, β_l, β_u are constants (depend on dimensionality, but not const. geometry).

MORE PRIOR RESULTS

- SER(noise power):
  - convex at high SNR
  - concave at low
  - odd number of inflex. points in-between
- Fading channels: average SER is
  - convex in low dimensions (mild condition)
  - "*Fading is Never Good*" in low dimensions
  - convex at high SNR (Rayleigh/Rice/polynomial at 0)

NUMEROUS ADVANTAGES/APPLICATIONS

- Strong analytical structure of convex problems[3]
- Efficient numerical algorithms, convergence, global solution
- Examples[4] (many more in the literature):
  - BLAST optimization
  - Time/power sharing in decrease/increase SER
  - n=1,2: “fading is never good”
  - convexity: power/time sharing is no good (Tx)
  - concavity: power/time sharing is no good (jammer)

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[3] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[4] S. Loyka, V. Kostina, F. Gagnon, Error Rates of the Maximum-Likelihood Detector for Arbitrary Constellations: Convex/Concave Behavior and Applications, IEEE Trans. Info. Theory, 2009, accepted.

NEW RESULTS: PEP IS CONVEX AT HIGH SNR

Theorem 3:

- Any n: the pairwise error probability (PEP) $\Pr\{\mathbf{s}_i \to \mathbf{s}_j\}$ is *convex at high SNR*, $\gamma \geq (n+\sqrt{2n})/d_{\min,i}^2$,
- $n=1,2$: *concave at low SNR*, $\gamma \leq (n+\sqrt{2n})/(d_{ij}+d_{\max,j})^2$, inflections (odd) in-between,
- $n>2$: *convex at low SNR*, $\gamma \leq (n-\sqrt{2n})/(d_{ij}+d_{\max,j})^2$, inflections (even) in-between.

MAIN RESULT: BER IS CONVEX AT HIGH SNR

Theorem 4:

- For *any* constellation and bit mapping (also coding), *BER is convex at high SNR* (under ML detection),

$$\mathrm{SNR} \geq \frac{n+\sqrt{2n}}{d_{\min}^2} \rightarrow \frac{d^2\mathrm{BER}}{d\mathrm{SNR}^2} > 0$$

CONVEXITY OF PEP/BER IN NOISE POWER

Theorem 5:

- the PEP is convex in noise power at high SNR(low noise).

Corollary 5.1:

- For *any* constellation and bit mapping (also coding), *BER is convex at high SNR*,

$$\mathrm{SNR} = \frac{1}{\sigma_0^2} \geq \frac{n+2+\sqrt{2(n+2)}}{d_{\min}^2} \rightarrow \frac{d^2\mathrm{BER}}{d\left(\sigma_0^2\right)^2} > 0$$

CONCLUSIONS

- **Importance of convexity/concavity**
- **SER is convex in low dimensions (n=1,2) or high SNR**
  - o dimensionality is critical, but not geometry
  - o in terms of both SNR and noise power
  - o universal bounds on derivatives
- **BER/PEP is convex at high SNR**
  - o any constellation/bit mapping/coding
  - o in SNR and noise power
- **Various applications/advantages**
  - o “fading is never good” in low dimensions