On Finite-SNR Diversity-Multiplexing Tradeoff

Sergey Loyka, George Levin
School of Information Technology and Engineering, University of Ottawa,
161 Louis Pasteur, Ottawa, Canada, K1N 6N5
E-mail: sergey.loyka@ieee.org

Abstract— Diversity–multiplexing tradeoff (DMT) presents a compact framework to compare various MIMO systems and channels in terms of the two main advantages they provide (i.e. high data rate and/or low error rate). This tradeoff was characterized asymptotically (SNR-> infinity) for i.i.d. Rayleigh fading channel by Zheng and Tse [1]. The SNR-asymptotic DMT overestimates the finite-SNR one [2].

In this paper, using the recent results on the size-asymptotic (in the number of antennas) outage capacity distribution, we derive and analyze the finite-SNR DMT for a broad class of channels (not necessarily Rayleigh fading). Systems with unequal number of Tx and Rx antennas exhibit qualitatively-different behavior from those with equal number of antennas: while the size-asymptotic DMT of the latter converges to the SNR-asymptotic DMT as the SNR grows, that of the former does not. However, the size-asymptotic DMT does provide an accurate approximation of the true DMT at low to moderately-high SNR, even for modest number of antennas, and hence is complementary of the SNR-asymptotic DMT of Zheng and Tse. Combining these two, a new DMT is obtained that is accurate over the whole SNR range.

A number of generic properties of the DMT that hold at any SNR, for any number of antennas (i.e. not only asymptotic, either in size or in SNR) and for any fading channel are given. In particular, we demonstrate that the linear interpolation of the DMT for fractional multiplexing gain in [1] does not hold at finite SNR.

Extensive Monte-Carlo simulations validate the analysis and the conclusions.

I. Introduction

Multi-antenna (MIMO) systems are able to provide either high spectral efficiency (spatial multiplexing, characterized by the multiplexing gain) or low error rate (high diversity, characterized by the diversity gain) via exploiting multiple degrees of freedom available in the channel. It was demonstrated by Zheng and Tse [1] that there is a fundamental tradeoff between these two gains, i.e. higher multiplexing gain can be achieved only at the expense of lower diversity gain and vice versa. Fundamentally, this is a tradeoff between the outage probability P_{out}, i.e. the probability that the fading channel is not able to support the transmission rate R, and the rate R, which can be expressed via the outage capacity distribution,

$$P_{out}(R) = \Pr\left[C < R\right] = F_C(R) \tag{1}$$

where C is the instantaneous channel capacity (i.e. capacity of a given channel realization), and $F_C(R)$ is its cumulative distribution function (CDF), also known as the outage capacity distribution. Defining the multiplexing gain r as

$$r = \lim_{\gamma\to\infty} R/\ln\gamma \tag{2}$$

where γ is the average SNR at the receiver, and the diversity gain as[1]

$$d = -\lim_{\gamma\to\infty} \frac{\ln P_{out}}{\ln\gamma} \tag{3}$$

the SNR-asymptotic ($\gamma\to\infty$) tradeoff for the independent identically distributed (i.i.d.) Rayleigh fading channel with the coherence time in symbols $l \ge m+n-1$ can be elegantly expressed as [1],

$$d(r) = (n-r)(m-r),\ r = 0,1,\ldots\min(m,n) \tag{4}$$

where m,n are the number of Tx, Rx antennas, for integer values of r, and using the linear interpolation in-between. The motivation for the definition of r in (2) is that the mean (ergodic) capacity $\overline{C}$ scales as $\min(m,n)\ln\gamma$ at high SNR,

$$\overline{C} \approx \min(m,n)\ln\gamma,\ \text{as } \gamma\to\infty \tag{5}$$

and the motivation for the definition of d in (3) is that P_{out} scales as γ^{-d} at high SNR,

$$P_{out} \approx c/\gamma^d,\ \text{as } \gamma\to\infty \tag{6}$$

where c is a constant independent of the SNR.

While this approach provides a significant insight into MIMO channels and also into performance of various systems that exploit such channels, it has a number of limitations. Specifically, it does not say anything about operational significance of r and d at realistic (finite) SNR. In other words, how high SNR is required to approach the asymptotes in (2),(3) with reasonable accuracy, so that, for example, d can be used to accurately estimate P_{out} using (6) and (4)? It was observed in [2], based on a lower bound to P_{out} for Rayleigh and Rician channels, that the finite-SNR diversity-multiplexing tradeoff (DMT) lies well below the curve in (4), so that proper modifications to the asymptotic results and definitions are required for realistic SNR values. Using the SNR-asymptotic ($\gamma\to\infty$) DMT to compare two systems may give incorrect results at low to moderate SNR.

To evaluate the DMT for arbitrary SNR, one would need to known the outage capacity distribution $F_C(R)$. While some results of this kind are available in the literature, their complexity prevents any analytical development. A number of compact analytical results have recently appeared on the outage capacity distribution of asymptotically large systems, i.e. when either $n\to\infty$ or $m\to\infty$, or both [3]-[7]. For a broad class of channels (under mild technical conditions), it turns out to be Gaussian with the mean and the variance determined by specifics of the channel [3]-[7].

[1] while the original definition in [1] employed the average error rate, since it is dominated by the outage probability, the definition in (3) is equivalent to it. This definition has also been adopted in [2].

These asymptotic results have been exploited in [8] to derive the diversity-multiplexing tradeoff for a broad class of channels (not just a Rayleigh or Rice fading one) and for realistic SNR values (including small and moderate ones), which we term here "size-asymptotic DMT" to distinguish it from the SNR-asymptotic DMT in (4). In particular, it was demonstrated that the finite-SNR tradeoff is below the SNR-asymptotic one in (4), and slowly approaches it as the SNR grows. While the convergence to the SNR asymptote in (4) is achieved at high but realistic SNR values (e.g. 20 dB) for smaller systems (e.g. 2x2), it is achieved at unrealistically high SNR (e.g. 80 dB) for larger systems (e.g. 10x10). On the other hand, the size-asymptotic capacity distributions result in compact closed-form approximations of the DMT at realistic SNR values, which are also sufficiently accurate for small systems (e.g. 2x2) [8].

In this paper, we continue the analysis initiated in [8] and provide a number of additional results as follows:

- While the analysis in [8] was limited to the $m=n$ case, here we generalize it to $m \neq n$ and demonstrate that the size-asymptotic DMT has qualitatively different behavior in this case (it does not converge to the SNR-asymptotic DMT when $\gamma \to \infty$). Due to slower convergence of the capacity distribution to the size-asymptotic one, larger number of antennas is required for the approximations to be accurate compared to the $m=n$ case.
- It is demonstrated that the size-asymptotic DMT is complementary to the SNR-asymptotic one, i.e. while the former is accurate at high SNR, the latter is accurate at low to intermediate SNR. Combining these two, one obtains DMT that is accurate at the whole SNR range.
- We derive a number of generic properties of the DMT (i.e. its monotonically decreasing nature, convexity, etc.) that hold for any fading channels (even for the channels for which the asymptotic results in [3]-[7] do not hold), any number of antennas and for arbitrary SNR, and, in particular,
- We demonstrate that the linear interpolation in (4) for fractional r is the consequence of $\gamma \to \infty$ and does not hold at finite SNR.

The rest of the paper is organized as follows. In section II we introduce the basic system model, various assumptions and briefly review the asymptotic outage capacity distributions (Theorem 1). Section III briefly reviews the main results in [8]. Section IV gives a number of generic properties of the DMT curve. In Section V, we rely on the asymptotic outage capacity distribution and generalize the results in [8] to the $m \neq n$ case, and validate the analysis and the conclusions using extensive Monte-Carlo simulations.`

II. SYSTEM MODEL AND OUTAGE CAPACITY DISTRIBUTION

The standard baseband discrete-time system model is adopted here,

$$\mathbf{r} = \mathbf{Hs} + \xi \tag{7}$$

where $\mathbf{s}$ and $\mathbf{r}$ are the Tx and Rx vectors correspondingly, $\mathbf{H}$ is the $n \times m$ channel matrix, i.e. the matrix of the complex channel gains between each Tx and each Rx antenna, and ξ is the additive white Gaussian noise (AWGN), which is assumed to be $\mathcal{CN}(\mathbf{0}, \sigma_0^2 \mathbf{I})$, i.e. independent and identically distributed (i.i.d.) in each branch. The assumptions on the distribution of $\mathbf{H}$ follow those of Theorem 1: the entries of $\mathbf{H}$ are assumed to be i.i.d. but otherwise arbitrary fading (this includes Rayleigh fading as a special case) [7], which can also be extended to correlated identically distributed and independent non-identically distributed entries [11], and to the correlated keyhole channel [5][6] (due to the page limit, the last two are not considered in this paper).

When full channel state information (CSI) is available at the Rx end but no CSI at the Tx end, the instantaneous channel capacity (i.e. the capacity of a given channel realization $\mathbf{H}$) in nats/s/Hz is given by the celebrated log-det formula [9],

$$C = \ln \det \left(\mathbf{I} + \frac{\gamma}{m} \mathbf{HH}^+ \right) \tag{8}$$

where γ is the average SNR per Rx antenna (contributed by all Tx antennas), "$^+$" denotes conjugate transpose.

For large m,n, the distribution of C takes on a remarkably simple form in a number of cases[2]:

Theorem 1 [[7], Theorem 2.76]: Let $\mathbf{H}$ be an $n \times m$ channel matrix whose entries are i.i.d. zero mean random variables with unit variance such that $E[|H_{ij}|^4] = 2$. As both $m,n \to \infty$ and $\beta = m/n$ is a constant, the instantaneous capacity in (8) is asymptotically (in m,n) Gaussian, with the following mean $\overline{C}$ and variance σ_C^2:

$$\begin{aligned} \frac{\overline{C}}{n} = \beta \ln\left(1 + \frac{\gamma}{\beta} - \frac{1}{4} F\left(\frac{\gamma}{\beta}, \beta\right)\right) + \ln\left(1 + \gamma - \frac{1}{4} F(\frac{\gamma}{\beta}, \beta)\right) \\ - \frac{\beta}{4\gamma} F\left(\frac{\gamma}{\beta}, \beta\right) \end{aligned} \tag{9}$$

$$\sigma_C^2 = -\ln\left(1 - \beta\left[\frac{1}{4\gamma} F\left(\frac{\gamma}{\beta}, \beta\right)\right]^2\right) \tag{10}$$

where $F(x,z) = (\sqrt{x(1+\sqrt{z})^2 + 1} - \sqrt{x(1-\sqrt{z})^2 + 1})^2$. At moderate to high SNR, (9) can be approximated as

$$\overline{C} \approx \min(m,n) \ln(\gamma / a) \tag{11}$$

where a is the high-SNR offset,

$$a = \begin{cases} e\beta(1-\beta)^{1/\beta - 1}, & \beta < 1 \\ e, & \beta = 1 \\ e(1-1/\beta)^{\beta - 1}, & \beta > 1 \end{cases}, \quad \sigma_C^2 \approx \begin{cases} -\ln(1-\beta), & \beta < 1 \\ \frac{1}{2} \ln \frac{\gamma}{4} + \frac{2}{\sqrt{\gamma}} \\ -\ln(1-1/\beta), & \beta > 1 \end{cases} \tag{12}$$

Note that Theorem 1 applies to a broad class of channels, not only Rayleigh or Ricean ones (as was the case in [1][2]).

Using the asymptotic distribution above, the outage probability can be expressed as

[2] Other asymptotic results are also available in the literature. However, we will rely only on this theorem in the present paper. Note that it can be extended to correlated channels as well [11].

$$P_{out}(R) = Q\left(\frac{\overline{C}-R}{\sigma_C}\right) \le \frac{1}{2}\exp\left(-\frac{1}{2}\left(\frac{\overline{C}-R}{\sigma_C}\right)^2\right) \quad (13)$$

where $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-t^2/2)dt$. The upper bound in (13) becomes tight at moderate to high SNR, so we use it as an approximation to P_{out} to simplify calculations.

III. FINITE-SNR DMT VIA ASYMPTOTIC CAPACITY DISTRIBUTION

In this section, we briefly summarize the main results in [8].

Finite-SNR DMT analysis requires using finite-SNR analogs of the definitions in (2),(3),

$$r = \frac{R}{\ln\gamma},\ d_\gamma = -\frac{\ln P_{out}}{\ln\gamma},\ \gamma > 1 \quad (14)$$

The convergence of the finite-SNR DMT to the asymptotic one in (4) is significantly improved if r is defined via $\overline{C}$, or via $\ln(\gamma/a)$, which is motivated by (11) and takes into account the high-SNR offset[3] a,

$$r = \frac{\min(m,n)R}{\overline{C}} \quad (15)$$

$$r = \frac{R}{\ln(\gamma/a)} \quad (16)$$

where (15) defines the rate as the $r/\min(m,n)$ fraction of the mean capacity or, equivalently, via the per-antenna mean capacity $\overline{C}/\min(m,n)$. Since (15) and (16) give similar results in terms of the DMT for $m \ne n$ (which is not a surprise, due to (11)), and since the definition in (15) has an important additional advantage (see Section V), we consider in details only the latter in this paper.

Another possible definition of d, which was introduced in [2], captures the differential effect of diversity, i.e. how much increase in SNR is required to decrease P_{out} by certain amount,

$$d'_\gamma = -\frac{\partial \ln P_{out}}{\partial \ln\gamma} \quad (17)$$

Note that the differential diversity gain d'_γ is insensitive to the constant c in (6) so that the convergence to the asymptotic value is faster. For high SNR, both definitions of the diversity gain (in (17) and (14)) give the same result.

Using (11) and (13) for the $m = n$ system and the multiplexing gain definition in (15), one obtains,

$$P_{out} \approx \frac{1}{2}\left(\frac{\gamma}{e}\right)^{-d(r)\Delta(\gamma)} \quad (18)$$

where $d(r) = (n-r)^2$ (as in (4)), and $\Delta(\gamma)$ quantifies the effect of finite SNR,

$$\Delta(\gamma) \approx 1 + 2/\left(\sqrt{\gamma}\ln(\gamma/e)\right) \quad (19)$$

Interpreting the $1/e$ term in (18) as a high-SNR offset (similarly to [10]), the diversity gain in (14) becomes $d_\gamma \approx d(r)\Delta(\gamma)$. Using (18), the differential diversity gain (17) can be expressed as

$$d'_\gamma = d(r)(\Delta(\gamma) + \gamma\ln(\gamma/e)\partial\Delta(\gamma)/\partial\gamma) \quad (20)$$

which, after some manipulations, can be simplified to

$$d'_\gamma \approx (n-r)^2\left(1 - \frac{1}{2\sqrt{\gamma}}\right) \quad (21)$$

Both d_γ and d'_γ converge to $d(r)$ in (4) as $\gamma \to \infty$, and the asymptote $d(r)$ provides good accuracy for $\gamma \ge 25 \approx 14dB$. If the multiplexing gain definitions in (14) and (16) are used instead, P_{out} exhibits anomalous behavior at small to moderate SNR (increases with SNR at certain interval), and the convergence to the asymptote takes place at substantially higher SNR for large systems: as an example, $\gamma \ge 120dB$ and $\gamma \ge 50dB$, respectively, for $n = 10$, $r = 9$. The reason for that is slow convergence of $\overline{C}$ to its high-SNR asymptote $n\ln\gamma$. Furthermore, the diversity gain alone cannot be used in this case to predict P_{out}, even at high SNR, since the constant c in (6) becomes anomalously high ($c \approx 10^4$ for $n = m = 10, r = 9$); on the contrary, c is moderate if (15) is used.

IV. GENERIC PROPERTIES OF THE DMT

We study below some generic properties of the finite-SNR DMT in (14), which hold for any fading channel and any SNR.

Proposition 1: The diversity gain $d_\gamma(r)$ is differentiable, monotonically decreasing function of the multiplexing gain r if the outage capacity distribution F_C is differentiable.

Proof: using the definitions in (14),

$$d_\gamma(r)' = \frac{\partial d_\gamma(r)}{\partial r} = -\frac{f_C(R)}{F_C(R)} \le 0 \quad (22)$$

where $F_C(R) = P_{out}$, and $f_C(R) = \partial F_C(R)/\partial R$ is the probability density (PDF) function of the instantaneous (outage) capacity C.

It immediately follows from Proposition 1 that $d_\gamma(r)$ cannot include linear interpolation of (4) since $d(r)$ in (4) is not differentiable at $r = \text{integer}$ (the left and right derivatives are not equal). Thus, we conclude that the linear interpolation in (4) is the consequence of $\gamma \to \infty$ and does not exists at any finite SNR when F_C is differentiable.

Proposition 2: The diversity gain $d_\gamma(r)$ attains its maximum at $r \to 0$ and its minimum at $r = r_{\max}$.

Proof: follows from (22) in Proposition 1.

Proposition 3: If $f_C(R)$ is unimodal[4], then $d_\gamma(r)$ is convex for $r \ge r_0 = R_0/\ln\gamma$,

$$d_\gamma(r)'' = \partial^2 d_\gamma(r)/\partial r^2 \ge 0 \quad (23)$$

Proof: follows from (22) and $\partial F_C/\partial r \ge 0$, $f_C{}' < 0, R > R_0$.

It follows from Propositions 1 and 3 that $d_\gamma(r)'$ is increasing for $r \ge r_0$ so that the loss in diversity gain due to

[3] [10] gives a detailed discussion of the importance of high-SNR offset in the capacity analysis of MIMO systems. Note that this offset is missing in (5).

[4] i.e. $f_C{}' = \partial f_C/\partial R > 0$ for $R < R_0$, $f_C{}' < 0$ for $R > R_0$, and $f_C{}' = 0$ for $R = R_0$.

increase in the multiplexing gain is smaller when the later is larger.

Proposition 4: If F_C or f_C is log-concave (e.g. Gaussian, Laplacian, exponential, etc.), then $d_\gamma(r)$ is convex, $d_\gamma(r)'' \geq 0$.

Proof:

$$d_\gamma(r)'' = -\ln\gamma \frac{\partial^2 \ln F_C}{\partial R^2} \geq 0 \quad (24)$$

where the inequality is due to log-concavity of F_C. If f_C is log-concave, then F_C is also log-concave, by the integration theorem of log-concave functions over a convex set [12].

It follows from these propositions that the generic form of the DMT function $d_\gamma(r)$ indicated in Fig. 1 holds true for a broad class of channels and at any SNR, and not only for Rayleigh and Rician ones as observed in [1][2]. Under the conditions of Proposition 3 or 4, the DMT curve $d_\gamma(r)$ lies below the line connecting the points $(0, d_{\max})$ and $(r_{\max}, d_{\min})$ (this follows from its convexity), it is monotonically decreasing, and its rate of decrease slows down for larger r. As an example, Fig. 2 shows the DMT curve of 2x2 Rayleigh fading channel at SNR=10 dB, which exhibits the properties in Propositions 1-4. Despite the small size, the size-asymptotic approximation works well when compared to Monte-Carlo simulations, and the approximation in (21) is of reasonable accuracy as well. Note that the linear interpolation of (4) is not present at the finite-SNR DMT curves.

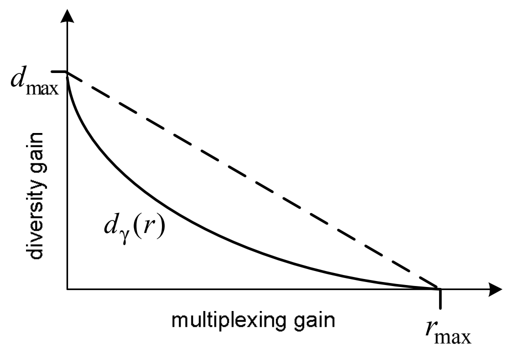

Fig. 1. Generic properties of the diversity-multiplexing tradeoff curve: $d_\gamma(r)$ is smooth (differentiable), monotonically decreasing and, for a broad class of channels, convex.

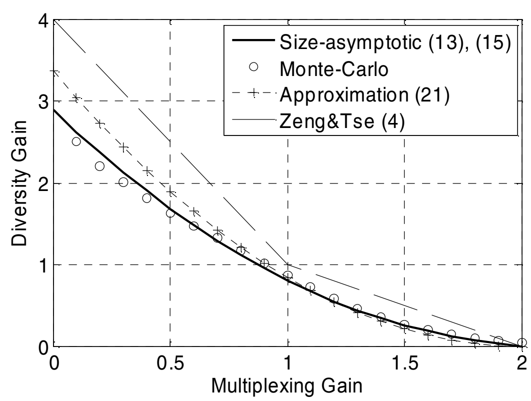

Fig. 2. Diversity-multiplexing tradeoff of 2x2 i.i.d. Rayleigh fading channel at SNR=10dB. The same generic properties of Fig. 1 are clearly observed.

V. DMT OF $m \neq n$ SYSTEMS

In this section, we generalize the results in [8] to the $m \neq n$ systems. Since we rely on Theorem 1, all the results here hold true for a broad class of channels for which Theorem 1 holds.

Using (11), (13) and (15), the outage probability can be expressed, after some manipulations, as

$$P_{out} \approx \frac{1}{2}\left(\frac{\gamma}{a}\right)^{-d_0(r)\Delta(\gamma)} \quad (25)$$

where

$$d_0(r) = (m^* - r)^2, \ \Delta(\gamma) \approx \frac{\ln(\gamma/a)}{-2\ln(1-\beta^*)} \quad (26)$$

and $m^* = \min(m,n)$, $\beta^* = \min(m,n)/\max(m,n)$. Using (17), after some manipulations, one obtains

$$d_\gamma' \approx 2d_0\Delta(\gamma) \approx (m^* - r)^2 \frac{\ln(\gamma/a)}{-\ln(1-\beta^*)} \quad (27)$$

The first term in (26), (27) is somewhat similar to the asymptotic ($\gamma \to \infty$) DMT of Zheng and Tse in (4), and the second term represents the effect of finite SNR. Note that, in agreement with Propositions 1-3, the diversity gain decreases monotonically with r, it attains its maximum and minimum at $r = 0$ and $r = r_{\max} = \min(m,n)$ respectively, and it is convex in r.

There are notable differences between the asymptotic ($\gamma \to \infty$) DMT of Zheng and Tse and those in (26), (27):

- Contrary to (4), neither d_γ nor d_γ' involve linear interpolation for fractional r. This re-enforces the conclusion after Proposition 1.
- The diversity gains in (26), (27) increase without bound as the SNR increases. This is in sharp contract to the SNR-asymptotic DMT (which is SNR-independent and finite) and also to the finite-SNR DMT for $m = n$ system in (18), (21), which converges to the asymptotic one as the SNR grows.

If the multiplexing gain definition in (14) is used, then P_{out} in (25) still holds, with

$$\Delta(\gamma) \approx \frac{1}{-2\ln(1-\beta^*)}\left(1 - \frac{r}{m^* - r}\frac{\ln a}{\ln(\gamma/a)}\right)^2 \ln\frac{\gamma}{a} \quad (28)$$

and

$$d_\gamma' \approx 2d_0\Delta(\gamma) \quad (29)$$

A note of caution is in order when (25)-(29) are used at high SNR, and especially if $\gamma \to \infty$, as in the original DMT definitions in [1]. The unbounded increase of the diversity gains in (26), (27), (29) when $\gamma \to \infty$ is the consequence of the Gaussian distribution in (13). However, since we apply this distribution here to a system with finite m,n, it serves only as an approximation. Furthermore, the accuracy of this approximation worsens as γ increases, i.e. when one moves to the distribution tail (small P_{out}), because, as it is well known from various formulations of the central limit theorem, convergence at distribution tails in terms of the relative accuracy is slower. Thus, we conclude that the approximations

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in (26), (27), (28) must “break down” at certain high SNR (low P_{out}), and the limiting transition $\gamma \to \infty$ is not legitimate.

This argument can be formalized as follows. For given finite m,n, the true capacity distribution F_C is approximated by the Gaussian P_{out} in (13) up to a certain accuracy ε, so that

$$\left|F_C - P_{out}\right| = \Delta F \le \varepsilon \quad (30)$$

For the asymptotically-based analysis to be meaningful at given true outage probability F_C, the normalized approximation error $\Delta F / F_C$ should be small,

$$\frac{\Delta F}{F_C} \le \frac{\varepsilon}{F_C} < 1 \quad (31)$$

from which one concludes that, for given approximation accuracy ε,

$$F_C > \varepsilon \quad (32)$$

for the Gaussian approximation to produce meaningful results. When F_C is a decreasing function of the SNR, $F_C(\gamma)$, the “meaningful” SNR range can be found from (32) as

$$\gamma < \gamma_{up} = F_C^{-1}(\varepsilon) \quad (33)$$

where $F_C^{-1}(\varepsilon)$ is the inverse function of $F_C(\gamma)$. As n,m increase, ε decreases and the upper bound γ_{up} increases. As $n,m \to \infty$, $\gamma_{up} \to \infty$ (due to the convergence to the asymptotic distribution, $\varepsilon \to 0$ as $n,m \to \infty$), but for any finite n,m, γ_{up} stays finite and the limiting transition $\gamma \to \infty$ is not legitimate, if the Gaussian approximation is used.

In fact, (33) determines the validity range of this approximation. Thus, we conclude that the size-asymptotic (in n,m) analysis of the DMT is accurate up to γ_{up}, while the SNR-asymptotic ($\gamma \to \infty$) DMT in (4) is accurate for $\gamma > \gamma_{ZT}$, where γ_{ZT} is some sufficiently high SNR (see section III and [8] for more details on the latter). Thus, our DMT analysis here is complementary to that of Zheng and Tse: the former applies where the later fails. Unfortunately, it is difficult to evaluate γ_{up}, γ_{ZT} analytically, and one has to resort to MC simulations. Based on these simulations, we observe that:

- d'_γ is monotonically increasing function of the SNR,
- it approaches $d(r)$ in (4) from below at high SNR.

Based on this, we propose the following combined estimation of the DMT that holds at any SNR,

$$d = \min\left\{d'_\gamma, d(r)\right\} \quad (34)$$

Extensive Monte-Carlo simulations have been carried out to validate the analysis and conclusions above. Some of the representative results are shown in Fig. 3-6.

Based on Fig. 3 and 4, we make the following observations:

- When r is defined as in (14), (16), P_{out} has anomalous behavior (increases with SNR) at small to moderate SNR, which is due to the fact that $R < \overline{C}$ on the corresponding interval but R increases faster than $\overline{C}$ with the SNR so that $\left|\overline{C} - R\right| / \sigma_C$ decreases; after the anomalous region this tendency is reversed. For the r definition via the mean capacity in (15), P_{out} behaves properly. This is similar to the $m = n$ case.
- The Gaussian approximation works for about $P_{out} \ge 10^{-2}$, which corresponds to $\gamma_{up} = 30dB$ for r defined in (15), (16), and to about $\gamma_{up} = 60dB$ for r defined in (14).
- High SNR offset ($c \approx 10^6$, see (6)) in P_{out} for $R = r \ln\gamma$ and $n = 10, m = 9$ makes it impossible to estimate P_{out} from the diversity gain alone, i.e. using $P_{out} \approx 1/\gamma^d$, no matter how high the SNR is . The rough estimation $P_{out} \approx 1/\gamma^d$ works only if c is on the order of unity. When this is not the case, c has to be accounted for as well. This indicates the limitation of the DMT, which ignores the constant c. Specifically, when two systems (or channels) are compared with the same r, and $d_1 > d_2$, it does not mean that system 1 performs better than system 2 in terms of P_{out} (or average error rate) since it may be that $c_1 > c_2$ and the latter effect is dominant. Hence, using the DMT curves alone to compare two systems may produce incorrect results, even at very high SNR. This suggests that the constant c (high-SNR offset) should also be included in the DMT if the error rate performance is of importance. This problem is somewhat eliminated by using the multiplexing gain definition in (16), as c becomes a moderate constant, but the anomalous behavior of the outage probability is not eliminated so that its estimation from the diversity gain alone at $\gamma \le 30dB$ is not possible. Using the definition in (15) results in more moderate offset $c \approx 10^2$ and the anomalous behavior of P_{out} disappears. For smaller systems (Fig. 4), this problem is not that severe as the SNR offset is much smaller, but the anomalous behavior of the outage probability at low to moderate SNR is still present for all definitions of the multiplexing gain but in (15).
- Comparing Fig. 3 and 4 to the similar ones in [8], we conclude that the convergence of P_{out} to the asymptotic one in (13) is slower for $m \ne n$, i.e. larger number of antennas is required for the same accuracy compared to the $m = n$ system. Alternatively, the Gaussian approximation “breaks down” at lower SNR when $m \ne n$.

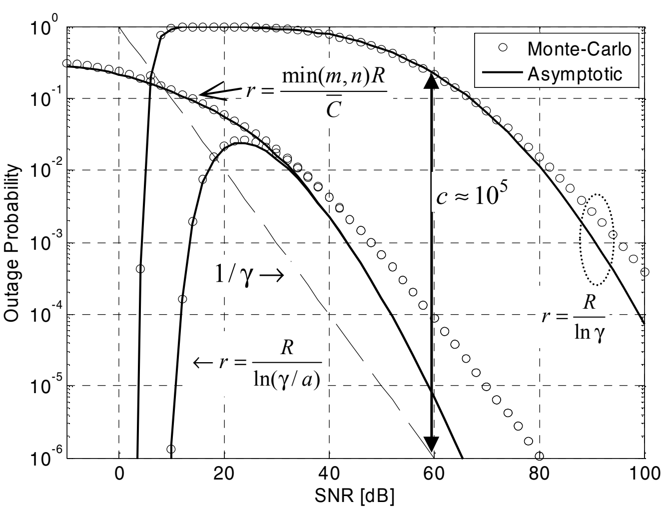

Fig. 3. Outage probability vs. SNR for various definitions of the multiplexing gain; $n = 10, m = 9, r = 8.5$; solid line – size-asymptotic from (9), (10), (13), circles – Monte-Carlo simulations (10^8 trials); dash line - $P_{out} = 1/\gamma$ (from (4), $d(r) = 1$). Note high SNR offset ($c \approx 10^5$).

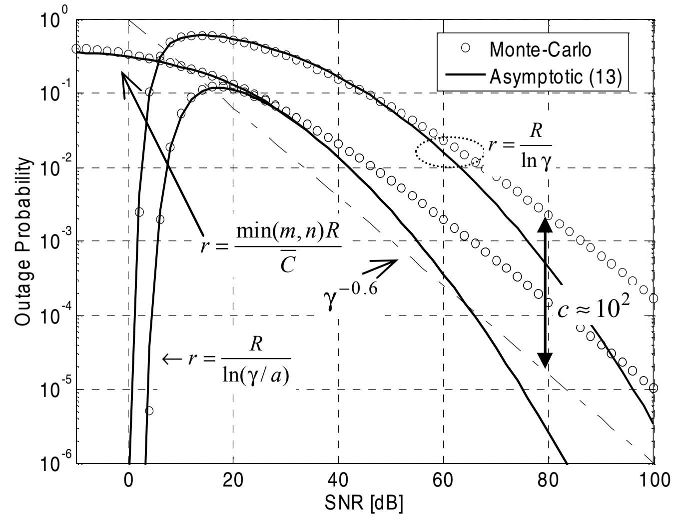

Fig. 4. Outage probability vs. SNR for various definitions of the multiplexing gain; $n = 3, m = 4, r = 2.7$; solid line – size-asymptotic from (9), (10), (13)**, circles – Monte-Carlo simulations (10^8 trials); dash line - $P_{out} = 1/\gamma^{0.6}$ (from** (4), $d(r) = 0.6$ **). Note smaller SNR offset in this case ($c \approx 10^2$).**

Based on Fig. 5 and 6, we observe that, as expected, the size-asymptotic DMT (based on Gaussian approximation in (13)) with the definition of r in (15) is accurate for $\gamma \le 40dB$ and $\gamma \le 30dB$, respectively. The SNR-asymptotic DMT in (4) with r definition in (15) becomes accurate at unreasonably high SNR, $\gamma \ge 80dB$, for large system (Fig. 4, Monte-Carlo simulations). For small system (Fig. 5), this range extends towards lower SNR, $\gamma \ge 50dB$, which is still very high. If r is defined via (14), this range moves to even higher SNR for large systems. On the contrary, the size-asymptotic DMT, including the approximation in (27), works at practical SNR range. Fig. 5 and 6 also re-enforce our earlier conclusion that the linear interpolation in (4) is the consequence of $\gamma \to \infty$ and does not show up at finite SNR. We also note that the combined DMT approximation in (34) is reasonably accurate at the whole SNR range.

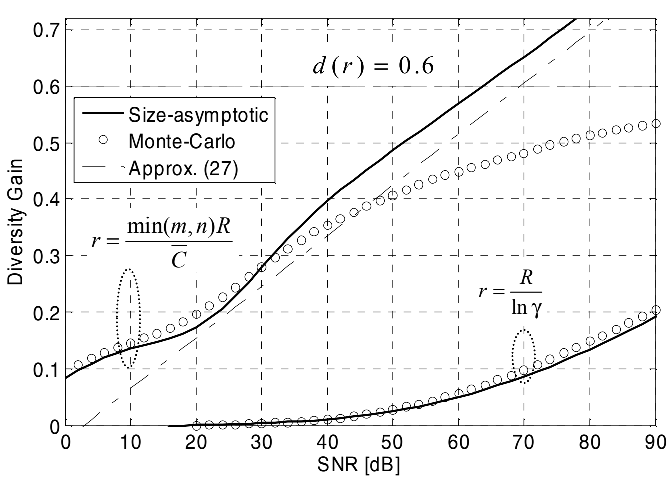

Fig. 5. Differential diversity gain vs. SNR for various definitions of the multiplexing gain; $n = 10, m = 9, r = 8.7$; solid line – size-asymptotic from (9), (10), (13)**, dashed – approximation in** (27)**.**

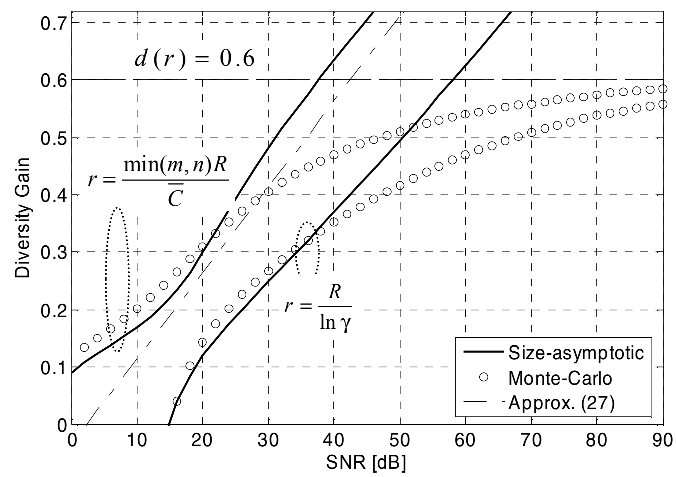

Fig. 6. Differential diversity gain vs. SNR for various definitions of the multiplexing gain; $n = 3, m = 4, r = 2.7$; solid line – size-asymptotic from (9), (10), (13)**, dashed – approximation in** (27)**.**

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