MULTI-KEYHOLES AND MEASURE OF CORRELATION IN MIMO CHANNELS

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ABSTRACT

Keyhole MIMO channels were predicted theoretically and also observed experimentally. However, they are not often encountered in practice since the assumption of a single propagation eigenmode is only a rough approximation of real propagation environments. This paper presents an extension to the single-keyhole channel model, termed a "multi-keyhole channel", which includes a number of statistically independent keyholes. Correlated full-rank and rank-deficient multi-keyhole channels are considered in detail. Under some general conditions the full-rank multi-keyhole channel is asymptotically Rayleigh fading if the number of keyholes is large. When the number of both Tx and Rx antennas is large, the capacity of a rank-deficient multi-keyhole channel is a sum of the capacities of the equivalent single-keyhole channels. The outage capacity distribution of both full-rank and rank-deficient multi-keyhole channels is asymptotically Gaussian. Based on the asymptotic capacity analysis, full ordering scalar measures of MIMO channel correlation and power imbalance are introduced.

Index Terms - MIMO system, keyhole channel, outage capacity, correlation.

1. INTRODUCTION

Multiple-Input-Multiple-Output (MIMO) systems have become an attractive solution in wireless communications due to enormously large spectral efficiency. One of the major statistical characteristics of a MIMO fading channel is its outage capacity, which gives the ultimate upper limit on the error-free information rate with a given probability of outage [1]. The outage capacity distribution of Rayleigh and Rice MIMO channels are well studied, and many analytical and empirical results are available. Chizhik et al [2] analytically predicted a keyhole channel, which can be modeled as a cascade of two Rayleigh fading channels separated by a single keyhole whose dimensions are much smaller than the wavelength. The presence of the keyhole degenerates the channel, i.e. its rank is one regardless the number of antennas [2]. Consequently, the capacity of such channels deteriorates significantly comparing to the Rayeigh channel with the same number of Tx and Rx antennas, even if the channel matrix entries are uncorrelated. Outage capacity distribution of single-keyhole channels is studied in [3]. Even though the single-keyhole channels may appear in some propagation scenarios, they are not often encountered in practice as the assumption of a single propagation eigenmode is only a rough approximation of real propagation scenarios [4]. Motivated by recent studies of the single-keyhole channel [3], we introduce a multi-keyhole channel to generalize and expand the range of applicability of the keyhole channel model. We establish, for the first time, a link between the keyhole and Rayleigh channels, and introduce a generic scalar measure of channel correlation and power imbalance in terms of their impact on the channel capacity. The significant advantage of this measure, as compared to that based on the majorization theory [5], is that any two channels can be compared without exceptions. Using this measure, we show analytically that both the correlation and the power imbalance have a negative impact on the asymptotic outage capacity.

2. SINGLE-KEYHOLE MIMO CHANNEL CAPACITY

Consider a spatially correlated single-keyhole MIMO channel with n_t Tx and n_r Rx antennas (see Fig. 1). Let the element H_{km}, $k = 1..n_r$; $m = 1...n_t$, of the channel transfer matrix $\mathbf{H}$ be a complex channel gain from the m-th transmit to the k-th receive antenna. The gain matrix of the keyhole channel is given by [2]

$$\mathbf{H} = \mathbf{h}_r \mathbf{h}_t^H \qquad (1)$$

where $(\cdot)^H$ denotes the Hermitian transpose, $\mathbf{h}_t$ $[n_t \times 1]$ and $\mathbf{h}_r$ $[n_r \times 1]$ are mutually independent random vectors representing the complex gains from the transmit antennas to the keyhole and from the keyhole to the receive antennas respectively. Assuming that the considered keyhole channel

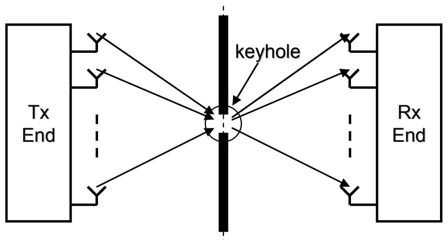

Fig. 1. A keyhole MIMO channel. Each end has rich multipath so that the sub-channels are correlated Rayleigh fading.

is a cascade of two correlated Rayleigh fading channels, $\mathbf{h}_t$ and $\mathbf{h}_r$ are complex circular symmetric correlated Gaussian vectors with correlation matrices $\mathbf{R}_t = E\{\mathbf{h}_t\mathbf{h}_t^H\}$ and $\mathbf{R}_r = E\{\mathbf{h}_r\mathbf{h}_r^H\}$ respectively, where $E\{\}$ denotes expectation. $\mathbf{H}$ is normalized so that $E\{\|\mathbf{H}\|^2\} = n_t n_r$, where $\|\ \|$ is the L_2 norm, and $n_t^{-1}E\{\|\mathbf{h}_t\|^2\} = n_r^{-1}E\{\|\mathbf{h}_r\|^2\} = 1$, which also implies $n_t^{-1}trace\{\mathbf{R}_t\} = n_r^{-1}trace\{\mathbf{R}_r\} = 1$.

From [1], when the channel state information (CSI) is available at the Rx but not the Tx end, the instantaneous capacity (i.e. the capacity for a given channel realization) of a quasi-static frequency flat MIMO channel in natural units $[nat]$ is given by:

$$C = \ln(\det[\mathbf{I} + \gamma_0 \mathbf{H}\mathbf{H}^H / n_t]) \tag{2}$$

where det is the determinant, $\mathbf{I}$ is $[n_r \times n_r]$ identity matrix and γ_0 is the average SNR per Rx antenna. The exact expression for the cumulative distribution functions (CDF) of C (the outage capacity distribution) when $\mathbf{R}_t$ or $\mathbf{R}_r$ are non-singular and have distinct eigenvalues is obtained in [3]. The following theorem is proven in [3]:

Theorem 1: When both n_t and n_r tend to infinity, the distribution of C is Gaussian in probability if $n_t^{-1}trace\{\mathbf{R}_t\} < \infty$, $n_t^{-2}\|\mathbf{R}_t\|^2 \to 0$ as $n_t \to \infty$, and $n_r^{-1}trace\{\mathbf{R}_r\} < \infty$, $n_r^{-2}\|\mathbf{R}_r\|^2 \to 0$ as $n_r \to \infty$. Moreover, if the channel is normalized so that $n_t^{-1}trace\{\mathbf{R}_t\} = 1$ and $n_r^{-1}trace\{\mathbf{R}_r\} = 1$, the asymptotic mean μ and the variance σ^2 of C are as follows:

$$\mu = \ln(1 + n_r\gamma_0); \quad \sigma^2 = n_t^{-2}\|\mathbf{R}_t\|^2 + n_r^{-2}\|\mathbf{R}_r\|^2 \tag{3}$$

From (3), the correlation affects σ^2 but not μ. Apparently, the higher σ^2, the smaller outage capacity a keyhole channel has at outage probabilities less than 0.5. The opposite is true at outage probabilities higher than 0.5 (however this range of outage probabilities has little importance from the practical point of view).

The next corollary follows immediately from Theorem 1.

Corollary 1: Asymptotically, the channel correlation enters into the outage capacity distribution through the norm only, i.e. even though two correlation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ (at either end) are different, they affect the capacity in the same way if $\|\mathbf{R}_1\| = \|\mathbf{R}_2\|$.

3. SCALAR MEASURES OF CORRELATION AND POWER IMBALANCE

Let $\mathbf{R}$ (either $\mathbf{R}_t$ or $\mathbf{R}_r$) belongs to $\Re$, where $\Re$ is a set of all $n \times n$ correlation matrices such that $trace(\mathbf{R}) = n$. Using the Cauchy-Schwarz inequality and the fact that every $\mathbf{R} \in \Re$ is positive semi-definite, it is straightforward to show that

$$n^{-1} \le n^{-2}\|\mathbf{R}\|^2 \le 1 \tag{4}$$

where the lower bound is achieved when the channel at the Tx(Rx) end is uncorrelated with the same power at each Tx(Rx) antenna, and the upper bound is achieved when the channel at Tx(Rx) end is fully correlated. Based on (4), there are two major effects that can increase $n^{-2}\|\mathbf{R}\|^2$: (i) non-uniform power distribution across the antennas (also termed power imbalance) and (ii) non-zero correlation. To analyze those effects separately, let us split $\mathbf{R} \in \Re$ into a sum of two matrices as follows:

$$\mathbf{R} = \mathbf{K} + \mathbf{P} \tag{5}$$

where $\mathbf{P} = diag\{\mathbf{R}\} - \mathbf{I}$ and $\mathbf{K} = \mathbf{R} - \mathbf{P}$; $diag\{\mathbf{R}\}$ is the diagonal matrix whose main diagonal is that of $\mathbf{R}$. Clearly, $\mathbf{P}$ and $\mathbf{K}$ account for the power imbalance and the correlation respectively. Since for any $\mathbf{R} \in \Re$, $trace(\mathbf{K}) = n$ and $trace(\mathbf{P}) = 0$, it is straightforward to show that the decomposition (5) is norm-orthogonal, i.e.

$$n^{-2}\|\mathbf{R}\|^2 = n^{-2}\|\mathbf{K}\|^2 + n^{-2}\|\mathbf{P}\|^2 \tag{6}$$

Moreover, it can be shown that

$$0 \le n^{-2}\|\mathbf{P}\|^2 \le 1 - n^{-1} \tag{7}$$

where the lower bound is achieved when all antennas have the same power (no power imbalance), and the upper bound is achieved when there is only one active Tx or Rx antenna. Furthermore, based on (4) and (6)

$$n^{-1} \le n^{-2}\|\mathbf{K}\|^2 \le 1 \tag{8}$$

where the lower bound is achieved when $\mathbf{K} = \mathbf{I}$ (uncorrelated channel), and the upper bound is achieved when the channel is fully correlated. Motivated by Corollary 1 and the discussion above, we introduce the following definitions.

Definition 1: A channel with correlation matrix $\mathbf{R}_1 \in \Re$ is said to be equally or more correlated than that with $\mathbf{R}_2 \in \Re$ if

$$\|\mathbf{K}_1\| \ge \|\mathbf{K}_2\| \tag{9}$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ correspond to $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively through (5). This scalar measure of the channel correlation is alternative to the measure given in [5], for channels with large n. Unlike [5], (9) is not based on the majorization theory and provides a complete correlation characterization with no exception (i.e. any two $\mathbf{R}_1, \mathbf{R}_2 \in \Re$ can be compared; see also the remark to the Definition 1 in [5]).

Definition 2: A channel with correlation matrix $\mathbf{R}_1 \in \Re$ has the same or more non-uniform power distribution (power imbalance) across antennas than that with $\mathbf{R}_2 \in \Re$ if

$$\|\mathbf{P}_1\| \geq \|\mathbf{P}_2\| \tag{10}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ correspond to $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively through (5).

To get some insight, consider a simple geometrical interpretation of Definitions 1 and 2 shown in Fig. 2. It follows that $n^{-2}\|\mathbf{R}\|^2$ is a mapping of $\Re$ onto a circle sector (a shadow region in Fig. 2). The channel correlation matrix $\mathbf{R}$ is represented by vector $\overline{R}$ such that

$$\left|\overline{R}\right| = n^{-1}\|\mathbf{R}\|; \ \ angle\{\overline{R}\} = \tan^{-1}\{\|\mathbf{P}\|/\|\mathbf{K}\|\} \tag{11}$$

Following Corollary 1, the asymptotic outage capacity is affected by the length of $\overline{R}$ but not by its angle. Consider two channels with correlation matrices represented by the vectors $\overline{R}_1$ and $\overline{R}_2$ such that $\left|\overline{R}_1\right| = \left|\overline{R}_2\right| = \left|\overline{R}\right|$ (see Fig. 2). Following Definitions 1 and 2, the channel with $\overline{R}_1$ is more correlated than one with $\overline{R}_2$. In contrast, the channel with $\overline{R}_2$ has more power imbalance across antennas. Nonetheless, the outage capacity of both channels is same. Therefore, the power imbalance and correlation between antennas have the same impact on the asymptotic capacity distribution of a single-keyhole channel if $\left|\overline{R}_1\right| = \left|\overline{R}_2\right|$.

As an example, consider, $n \times n$ exponential correlation matrix model for $\mathbf{R}$ [6]. The following holds true as $n \to \infty$ [3],

$$n^{-2}\|\mathbf{K}\|^2 \to \frac{1}{n} \cdot \frac{1+|r|^2}{1-|r|^2} \to 0; \ |r| < 1 \tag{12}$$

where r is a complex correlation parameter. From (12), the measure of correlation converges to zero (as required by Theorem 1) and increases monotonically with $|r|$. The latter fact supports Definition 1.

4. MULTI-KEYHOLE CHANNEL

Following [4], the ideal single-keyhole channel is not often encountered in practice since the assumption of a single non-zero eigenmode is only a rough approximation of real propagation scenarios. More often, the channel may have a number of keyholes. By extending (1), the channel transfer matrix of such a multi-keyhole channel can be represented as:

$$\mathbf{H} = \sum\nolimits_{k=1}^{M} a_k \mathbf{h}_{rk}\mathbf{h}_{tk}^H = \mathbf{H}_r \mathbf{A} \mathbf{H}_t^H \tag{13}$$

where M is a number of keyholes, a_k is the complex gain of the *k*-th keyhole, $\mathbf{h}_{tk}$ $[n_t \times 1]$ and $\mathbf{h}_{rk}$ $[n_r \times 1]$ are random vectors representing the complex gains from the transmit antennas to the *k*-th keyhole and from the *k*-th keyhole to the receive antennas respectively; $\mathbf{H}_t = [\mathbf{h}_{t1}..\mathbf{h}_{tM}]$, $\mathbf{H}_r = [\mathbf{h}_{r1}..\mathbf{h}_{rM}]$ are $[n_t \times M]$ and $[n_r \times M]$ matrices respectively, and $\mathbf{A}$ is an $[M \times M]$ diagonal matrix with elements $\mathbf{A}_{kk} = a_k$, $k = 1..M$. Suppose that for every k, $\mathbf{h}_{tk}$ and $\mathbf{h}_{rk}$ are mutually independent complex circular symmetric Gaussian vectors with correlation matrices $\mathbf{R}_{tk} = E\{\mathbf{h}_{tk}\mathbf{h}_{tk}^H\}$ and $\mathbf{R}_{rk} = E\{\mathbf{h}_{rk}\mathbf{h}_{rk}^H\}$. Suppose also, that the keyholes are independent of each other, i.e $E\{\mathbf{h}_{tk}\mathbf{h}_{tm}^H\} = E\{\mathbf{h}_{rk}\mathbf{h}_{rm}^H\} = \mathbf{0}$ for any $k \neq m$. For comparison purposes, $\mathbf{H}$ is normalized so that $E\{\|\mathbf{H}\|^2\} = n_t n_r$ and for every k, $n_t^{-1}E\{\|\mathbf{h}_{tk}\|^2\} = n_r^{-1}E\{\|\mathbf{h}_{rk}\|^2\} = 1$, which implies

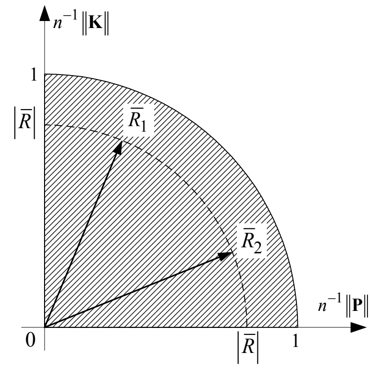

Fig. 2. Geometrical interpretation of the power imbalance and correlation effects.

$$\sum\nolimits_{k=1}^{M} |a_k|^2 = 1 \tag{14}$$

i.e. the average SNR per Rx antenna is constant regardless of the number of keyholes. Substituting (13) in (2), the instantaneous capacity of a frequency flat quasi-static multi-keyhole MIMO channel in natural units $[nat]$ with the CSI available at the Rx end only is given by:

$$C = \ln\left(\det[\mathbf{I} + \gamma_0 n_r \mathbf{B}_r \mathbf{A} \mathbf{B}_t \mathbf{A}^H]\right) \tag{15}$$

where $\mathbf{B}_t = \mathbf{H}_t^H \mathbf{H}_t / n_t$ and $\mathbf{B}_r = \mathbf{H}_r^H \mathbf{H}_r / n_r$. Below, we consider two types of the multi-keyhole MIMO channels:

Full-Rank Multi-Keyhole Channel ($M \geq \min\{n_t, n_r\}$):

Theorem 2: A full-rank multi-keyhole channel is asymptotically Rayleigh fading as $M \to \infty$ if

$$\lim_{M\to\infty} \max_k \{|a_k|^2\} = 0, \tag{16}$$

i.e. the power contribution of each single keyhole approaches zero as M goes to infinity. A proof of Theorem 2 follows directly from the Lindeberg-Feller Theorem [7] and omitted for brevity. Fig 3 compares the outage capacity distributions of a 2x2 multi-keyhole channel with $|a_k| = \sqrt{1/M}$ and of the equivalent Rayleigh channel. The correlation at both Tx and Rx ends and for all M keyholes is represented by the exponential model [6] with $|r| = 0.5$. The Kronecker model [8] was used to simulate the correlation at Tx and Rx ends in the Rayleigh channel. Clearly, the outage probability of the multi-keyhole channel

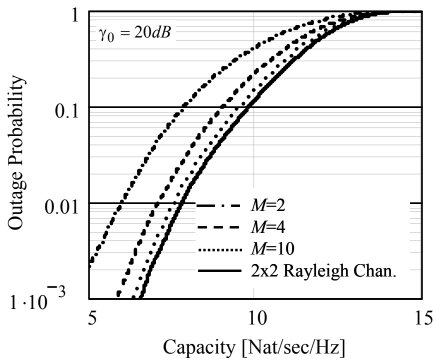

Fig. 3. Outage capacity distribution of 2x2 full-rank multi-keyhole channel vs. the number of keyholes M.

decreases with M and becomes close to that of the equivalent Rayleigh channel already for $M = 10$.

Rank-Deficient Multi-Keyhole Channel ($M < \min\{n_t, n_r\}$):
Theorem 3: Let C be an instantaneous capacity of the multi-keyhole channel (15). If for every $k, m = 1..M$, $n_t^{-1} trace\{\mathbf{R}_{tk}\} < \infty$, $n_t^{-2} trace[\mathbf{R}_{tk}^H \mathbf{R}_{tm}] \to 0$ as $n_t \to \infty$, and $n_r^{-1} trace\{\mathbf{R}_{rk}\} < \infty$, $n_r^{-2} trace[\mathbf{R}_{rk}^H \mathbf{R}_{rm}] \to 0$ as $n_r \to \infty$, the following holds true as both n_t and n_r go to infinity:

$$C \overset{p}{\to} \sum_{k=1}^{M} \ln\left(1 + |a_k|^2 \gamma_0 \|\mathbf{h}_{tk}\|^2 \|\mathbf{h}_{rk}\|^2 / n_t\right) \qquad (17)$$

where $\overset{p}{\to}$ means convergence in probability. Hence, the asymptotic instantaneous capacity of a rank-deficient multi-keyhole channel is the sum of the capacities of the equivalent single-keyhole channels. A proof is omitted due to the page limit. Note that the conditions of Theorem 1 for each single keyhole follow from those of Theorem 3; therefore the terms of the sum (17) are Gaussian. Moreover, they are independent as the keyholes are assumed to be independent. Therefore, similarly to the single-keyhole channel, the asymptotic instantaneous capacity of the multi-keyhole channel is also Gaussian with the mean μ and the variance σ^2 given as follows:

$$\begin{aligned} \mu &= \sum_{k=1}^{M} \ln\left(1 + |a_k|^2 \gamma_0 n_r\right) \\ \sigma^2 &= n_t^{-2} \sum_{k=1}^{M} \|\mathbf{R}_{tk}\|^2 + n_r^{-2} \sum_{k=1}^{M} \|\mathbf{R}_{rk}\|^2 \end{aligned} \qquad (18)$$

It follows from comparing (18) to (3) that the decomposition into the two orthogonal effects (the correlation and the power imbalance, see (5)) holds true not only for a single-keyhole channel, but also for the rank-deficient multi-keyhole channels with an arbitrary number of keyholes. Therefore, the impact of correlation and power imbalance on the asymptotic outage capacity distribution of the single-keyhole channel and of the rank-deficient multi-keyhole channel are same, i.e. the capacity at outage probabilities less than 0.5 decreases with correlation or/and power imbalance across the antennas.

5. CONCLUSION

Motivated by recent studies of the single-keyhole channel, a multi-keyhole channel model is introduced and investigated. This model establishes a link between the keyhole and Rayleigh MIMO channels. As a byproduct of the present study, a new scalar measure of correlation and power imbalance is introduced. This measure allows complete rather than partial ordering of the channels and can also be applied to other MIMO channels, whose capacity depends on the norm of correlation matrices, for example, to correlated Rayleigh channels [9]. The fact that the outage capacity distribution of all considered channels is asymptotically Gaussian may indicate that the Gaussian distribution has a certain degree of universality in the outage capacity analysis of MIMO channels.

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