Outage Probability of the BLAST Algorithm in a Rayleigh Fading Channel

S. Loyka[1], F. Gagnon[2]

Abstract— Analytical approach to outage and BER analysis of the V-BLAST and D-BLAST algorithms in i.i.d. Rayleigh channel is presented in this paper. Based on the previous results, which allow an exact analytical BER and outage analysis of a 2xn system in a closed-form, generic case of mxn system is analytically analyzed. While closed-form exact analysis is not feasible, a tight bound for the outage probability is derived and validated using Monte-Carlo simulations. It is shown that the optimal ordering procedure results at a SNR gain of m at the 1st step for $m \le 4$. We conjecture that this is true for larger m as well. The results above are also extended to the case of a D-BLAST system.

I. INTRODUCTION

High capacity promise of the MIMO architecture over multipath channels can be achieved using appropriate signal processing algorithm. The BLAST algorithm is a promising solution due to its low implementation complexity [1]. Some analysis of its performance has been done, mainly using numerical techniques (Monte-Carlo). The analytical analysis is a challenging problem. Very limited progress has been done so far.

The purpose of this paper is to present an analytical approach to bit error rate (BER) and outage probability analysis of the BLAST in a Rayleigh fading channel. The proposed approach results in closed-form exact analytical expressions for the outage and BER of the 2xn systems (i.e., with 2 Tx and n Rx antennas), which are fully validated using extensive Monte-Carlo simulations. At the moment, we are not able to derive exact closed-form expressions for the generic case of mxn system. However, we present some tight bounds, obtain closed-form expressions based on these bounds and demonstrate, through Monte-Carlo simulations, that the bounds/approximate expressions capture many essential features of the system performance. Finally, based on the results obtained for the V-BLAST, we demonstrate that similar results follow for the D-BLAST.

II. V-BLAST ALGORITHM ANALYSIS: 2XN SYSTEM

The V-BLAST algorithm has been discussed in details elsewhere [1]. The main idea of the BLAST is to split the information bit stream into several sub-streams and transmit them in parallel using a set of Tx antennas (the number of Tx antennas equals the number of sub-streams) at the same time and frequency. At the Rx side, each Rx antennas "sees" all the transmitted signals, which are mixed due to the nature of the wireless propagation channel. Using appropriate signal processing at the Rx side, these signals can be unmixed so that the matrix wireless channel is transformed into a set of virtual parallel independent channels (provided that mutltipath is rich enough).

The V-BLAST processing begins with the 1st Tx symbol and proceeds in sequence to the m-th symbol. When the optimal ordering procedure is employed, the Tx indexing is changed prior to the processing. The main steps of the V-BLAST processing (detection) algorithm are as follows [1]:

1. *The interference cancellation step*: at the i-th processing step (i.e., when the signal from the i-th transmitter is detected) the interference from the first i-1 transmitters can be subtracted based on the estimations of the Tx symbols (which are actually assumed to be error-free) and the knowledge of the channel matrix **H**.
2. *The interference nulling step*: based on the knowledge of the channel matrix, the interference from yet-to-be-detected symbols can be nulled out using the Gramm-Schmidt orthogonalization process (applied to the column vectors of **H**).
3. *The optimal ordering procedure*: the order of symbol processing is organized according to their after-processing SNRs in the decreasing order (i.e., the symbol with highest SNR is detected first).

Geometrical framework for the closed-form analysis of the algorithm operation has been discussed in details in [2,3]. Based on those results, the signal fading in the V-BLAST system can be analyzed. In particular, we consider the outage probabilities (i.e., the probability that the after-detection signal power is less than the specified value) and diversity order (i.e., the asymptotic slope of the outage probability curve). Assuming uncorrelated quasi-static frequency-flat Rayleigh fading channel and ignoring the optimal ordering procedure, it can be proved that the diversity order at the *i*-th step is (*n-m+i*). Hence, the maximum ratio combining (MRC) outage probability of appropriate order can be used. Consequently, all the results for BER of MRC hold true in the case of V-BLAST as well.

However, the optimal ordering procedure "mixes" things up and makes the analysis more challenging. Fortunately, the approach above can be extended to account for the optimal ordering [2,3]. For the 2xn system, it can be shown that the "after-processing" channel power gain s_1 at 1st detection step is distributed as follows,

$$P_1(x) = \Pr\left[s_1 < x\right] = \int_0^{\pi/2} F_n^2\left(\frac{x}{\sin^2\varphi}\right) f_\varphi(\varphi)\, d\varphi \qquad (1)$$

where $F_n(x) = 1 - e^{-x}\sum_{k=0}^{n-1} x^k/k!$ is n-th order MRC distribution, $f_\varphi(\varphi) = 2(n-1)(\sin\varphi)^{2n-3}\cos\varphi$ is the PDF of the angle between two column vectors of the channel matrix. (1) can be further reduced to the following closed-form expression: $P_1(x) = 1 - p_1(x)e^{-x} + p_2(x)e^{-2x}$, where $p_1(x)$ and $p_2(x)$ are polynomials of degree at most *(n-2)* and *(2n-3)* correspondingly (see [2,3] for the details). The asymptotic behavior of the outage probability can be shown to be

$$P_1(x) \approx \left(x/2\right)^{n-1}/(n-1)!, \quad x \to 0 \qquad (2)$$

Comparing it with (n-1)-order MRC asymptotic behavior, $F_{MRC}(x) \approx x^{n-1}/(n-1)!$, we conclude that the effect of the optimal ordering at the 1st detection step is to increase SNR by 3 dB rather than to increase the diversity order.

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[1] School of Information Technology and Engineering, University of Ottawa, 161 Louis Pasteur, Ottawa, Ontario, Canada, K1N 6N5 (e-mail: sergey.loyka@ieee.org)

[2] Department of Electrical Engineering, Ecole de Technologie Superieure, 1100, Notre-Dame St. West, Montreal (Quebec), H3C 1K3, Canada

The conditional outage probability at the 2nd detection step (conditioned on no detection error at the 1st step) is given by $P_2(x) = \Pr[s_2 < x] = F_n(x)[2 - F_n(x)]$. The effect of optimal ordering at the 2nd detection step is to increase the outage probability twice. This is the "price" to pay for the increased SNR at the 1st step. Note that these are *exact* expressions for the outage probabilities.

Using the outage probabilities above, average BER can be found at each detection step for various modulation formats in a straightforward way (note that only conditional outage probability at the 2nd step is required to find the total average BER). Detailed results, including exact closed-form expressions for the average BER, are presented in [2,3] and omitted here due to the lack of space. It can be shown that for moderate to high SNR, 1st step BER is dominant and the effect of error propagation is negligible (i.e., second-order one).

III. V-BLAST Algorithm Analysis: mxn System

We now generalized the previous results to the case of mxn system. The generalization is highly non-trivial and presents serious difficulties, which we resolve at the moment using various bounds and approximations.

1st step outage probability can be found in this case using a generalized from of (1):

$$P_1(x) = \int_0^{\pi/2} \ldots \int f_\varphi(\varphi_1,\ldots\varphi_m) \prod_{i=1}^{m} F_n\left(\frac{x}{\sin^2\varphi_i}\right) d\varphi_i \qquad (3)$$

$$f_\varphi(\varphi) = 2(m-1)C_{n-1}^{m-1} \sin^{2(n-m)+1}\varphi \cdot \cos^{2m-3}\varphi \qquad (4)$$

where $f_\varphi(\varphi_1,\ldots\varphi_m)$ is the joint pdf of $\{\varphi_1,\ldots\varphi_m\}$, φ_i being the angle between $\mathbf{h}_i$ (i-th column vector of $\mathbf{H}$) and the sub-space spanned all the other column vectors of $\mathbf{H}$, $f_\varphi(\varphi)$ is the marginal pdf, and C_n^m is the binomial coefficient. Note that $f_\varphi(\varphi_1,\ldots\varphi_m)$ is symmetric with respect to $\{\varphi_1,\ldots\varphi_m\}$ (any two angles can be exchanged without affecting the pdf). The angles are neither independent nor fully correlated, which makes it very difficult to find the joint pdf required in (3). To this end, we use the Holder inequality and derive the following bounds:

$$P_1(x) \le \int_0^{\pi/2} f_\varphi(\varphi) F_n^m\left(\frac{x}{\sin^2\varphi}\right) d\varphi \le \int_0^{\pi/2} f_\varphi(\varphi) F_n\left(\frac{x}{\sin^2\varphi}\right) d\varphi \quad (5)$$

While the 2nd bound is loose, the 1st one is, as we show later on, quite tight. Additionally, since $\{\varphi_1,\ldots\varphi_m\}$ are exchangeable random variables which are known to have non-negative correlation, it can be shown that

$$\left(\int_0^{\pi/2} f_\varphi(\varphi) F_n\left(\frac{x}{\sin^2\varphi}\right) d\varphi\right)^m \le P_1(x) \qquad (6)$$

After some manipulations, the 1st bound in (5) can be presented as:

$$B_1(x) = \beta\left\{\sum_{l=0}^{m} \alpha_l (J_{3l} + J_{4l}) + \sum_{l=2}^{m} \alpha_l J_{2l}\right\} \qquad (7)$$

where $\beta = (-1)^{m-2}(m-1)C_{n-1}^{m-1}$, $\alpha_l = (-1)^l C_m^l$,

$$J_{2l} = e^{-lx}(lx)^{n-m+1} \sum_{p=0}^{l(n-1)-n+m-2} a_{pl}(lx)^p$$

$$J_{3l} = (-1)^{n+1} e^{-lx}(-lx)^{n-m+1} \sum_{p=0}^{m-3} b_p(-lx)^p,\ J_{4l} = e^{-lx} \sum_{p=0}^{n-2} d_p(-lx)^p$$

$$a_{pl} = \sum_{k=\max[0,m-2-p]}^{m-2} \frac{(-1)^k C_{m-2}^k}{(p+k-m+2)!} \sum_{i=\max[0,p-m+2]}^{l(n-1)-n} c_{i+n,l}\, l^{-i-n}(k+i)!$$

$$b_p = \sum_{k=0}^{p} \frac{(-1)^k C_{m-2}^{k-p+m-2}}{k!} \sum_{i=0}^{m-p-3} \frac{(k+i)!}{(i+p+n-m+2)!}$$

$$d_p = \sum_{k=0}^{\min[m-2,n-2-p]} (-1)^k (n-k-p-2)!\, C_{m-2}^k \sum_{i=0}^{p} \frac{(-1)^i}{i!(n-k-i-1)!}$$

$$c_{i,l} = \sum_{i_1+\ldots+i_l=i} \frac{1}{i_1!\ldots i_l!},\ 0 \le i_1,\ldots i_l \le n-1$$

While the expression for a_{pl}, b_p, d_p may appear complicated, they can be evaluated in advanced (i.e., a table of coefficients is built for a given order of the system) and do not need to be changed during simulations. Note that the bound is presented as a product of exponents and polynomials of finite order (which depends on the system order) and, hence, the procedure is very efficient numerically. Note also that (7) reduces to the known case of $m = 2$, as it should be. To get some insight and to evaluate the bound accuracy, we further consider 3x3 system.

Outage of 3x3 V-BLAST: The 1st step outage is bounded as

$$\begin{aligned} P_1(x) \le B_1(x) = 1 - 3e^{-x} + e^{-2x}\left(3 + 15x/8 + 3x^2/8\right) \\ -e^{-3x}\left(1 + 110x/81 + 7x^2/9 + 2x^3/9 + x^4/36\right) \end{aligned} \qquad (8)$$

The asymptotic behavior of the bound is

$$B_1(x) \approx 335x/648 \approx x/2,\ x \to 0 \qquad (9)$$

which is the same as the asymptotic outage probability of the 2x2 system [2,3]. 1st order diversity and 3 dB gain due to optimal ordering are apparent (this 3 dB gain transforms asymptotically into 3 dB gain in terms of the average BER).

The 2nd step outage cannot be easily evaluated since the ordering procedure at the 1st step affects the channel statistics at the 2nd step. We evaluate the conditional outage probability at the 2nd step (i.e., conditioned on no detection error at the 1st step – this is what we need to evaluate the total outage probability and BER [3]). As an approximation, we assume that the channel statistics at the 2nd step is not affected by the optimal ordering at the 1st step (i.e., the channel coefficients are still i.i.d. complex Gaussian). Under this assumption, the 2nd step outage probability is the same as that of a 2x3 system at the 1st step (since the first bit stream has been detected and eliminated at the 1st step), whose outage is [2,3],

$$P_2(x) = 1 - 2e^{-x}(1+x) + e^{-2x}\left(1 + 2x + 9x^2/8 + x^3/4\right) \quad (10)$$

Its asymptotic behavior is

$$P_2(x) \approx x^2/8,\ \ x \to 0 \qquad (11)$$

Second-order diversity is obvious.

The 3rd step conditional outage probability can be evaluated in a similar way. Assuming no change in the channel statistics due to the ordering in the first two steps, it is the same as that of a 2x3 system at the second step,

$$P_3(x) = F_3(x)[2 - F_3(x)] \qquad (12)$$

Its asymptotic behavior is $P_3(x) \approx 2F_3(x) \approx x^3/3$, which indicates the 3rd order diversity.

Extensive Monte-Carlo simulations have been carried out to evaluate the accuracy of the bound and approximations involved. The results are shown in Fig. 1. Clearly, the 1st step bound is quite accurate (given its simple nature) and it underestimates the performance by 2 dB. The actual asymptotic behavior of the outage probability is

$$P_1(x) \approx x/3, \quad x \to 0 \tag{13}$$

We conjecture that in general, the asymptotic outage probability is

$$P_1(x) \approx x/m, \quad x \to 0 \tag{14}$$

Note that it is true for 2xn system [2,3], our simulations here confirm it for 3x3, 3x4 and 4x4 systems.

The 2nd step performance is overestimated by 3 dB. However, as fig. 2 demonstrates, it is predicted extremely well by the 2nd order MRC outage curve. We attribute this to the joint effect of two opposite factors: 1) performance loss at the 2nd step due to optimal ordering at the 1st (the same as for 2xn system), and 2) performance improvement due to the 2nd step optimal ordering. Apparently, this two effects compensate each other and the resulting outage is the same as that of 2nd order MRC.

The 3rd step performance is estimated quite accurately by the approximate expression (12) (within 1 dB). MRC outage curve would provide worse approximation in that case.

Outage of 4x4 V-BLAST: The validity of the approximations above is not limited to a 3x3 system. As an example, we use the same approximations to analyze 4x4 system. Fig. 2 shows the outage probability at first 3 steps. The 1st step bound and its asymptotic behavior are obtained using (7). The asymptotic behavior of the 1st step outage is given by (14) for m=4.

The 2nd step outage has been analytically estimated using the 1st step outage of a 3x4 system, which is within 1.5 dB of the actual performance. Note that it is not the same as MRC anymore. However, the 3rd order performance is virtually the same as that of 3rd order MRC. The analytic estimation of the performance (using 1st step outage of a 2x4 system) overestimates it by approximately 3 dB. We attribute this to the effect of the optimal ordering at the 1st and 2nd steps.

It should be noted that using the analytical approximations for the outage probabilities, the average BER can be evaluated in a straightforward way. Closed-form BER expressions can also be derived for various modulation formats (in the same way as in [2,3]).

IV. BER of the D-BLAST

Using the Bayes formula, the instantaneous unconditional BER at i-th step (i.e. including the error propagation from first (i-1) steps) is

$$\widetilde{P}_i = \sum_{j=1}^{i} P_j \prod_{k=1}^{j-1} (1-P_k), \tag{15}$$

where P_j is the conditional BER at j-th step. The product gives the probability of no error at first (*j*-1) steps, and the entire expression gives the probability of at least one error at first *i* steps (due to the error propagation, an error at any step form 1 to (*i*-1) will result at error at step *i*). Using this, the total unconditional BER of the D-BLAST is the same for all the Txs (due to the problem symmetry) and can be expressed as:

$$P_D = m^{-1} \sum_{i=1}^{m} \tilde{P}_i \tag{16}$$

It is instructive to consider the asymptotic behavior of (16), i.e. when the average SNR is high and the 1st step BER is dominant (recall that the diversity order increases with the step number), $P_1 >> P_2 ... >> P_m$. In this case, $P_D \approx P_1$. Clearly, the effect of antenna "rotation" is asymptotically negligible. However, at small SNR mode the D-BLAST does provide advantage over the V-BLAST,

$$P_D = m^{-1} \sum_{i=1}^{m} (m+1-i) \hat{P}_i < P_V = \sum_{i=1}^{m} \hat{P}_i, \tag{17}$$

where $\hat{P}_i = P_i \prod_{k=1}^{i-1} (1-P_k)$.

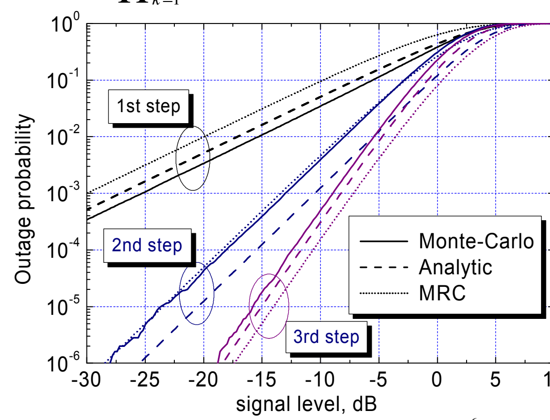

Fig. 1. Outage probabilities of 3x3 V-BLAST. $5*10^6$ trials have been used for Monte-Carlo simulations.

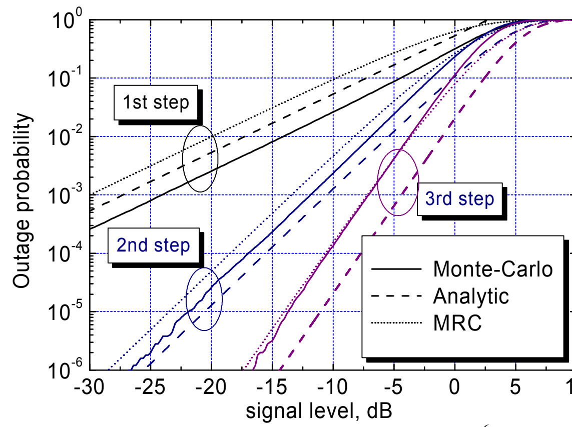

Fig. 2. Outage probabilities of 4x4 V-BLAST. $5*10^6$ trials have been used for Monte-Carlo simulations.

V. References

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