MULTI-ANTENNA CAPACITIES OF WAVEGUIDE AND CAVITY CHANNELS

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Abstract—*MIMO capacity of waveguide and cavity channels is analyzed in details in this paper. The rationale for this is three-fold: (i) waveguide / cavity models can be used to model corridors, tunnels and other confined space channels, (ii) this is a canonical problem; its analysis allows to develop appropriate techniques, which can be further used for more complex problems, (iii) it allows to shed light on the relation between information theory and electromagnetism and, in particular, to establish the limits imposed by the laws of electromagnetism on achievable channel capacity. It is demonstrated that the number of degrees of freedom of the electromagnetic field inside of waveguides, which can be used for MIMO communication, is determined by the waveguide cross-sectional area expressed in terms of the wavelength. A system architecture is proposed, which allows using these degrees of freedom.*

Keywords: *MIMO, channel capacity, waveguide*

I. INTRODUCTION

Multi-antenna systems (also known as MIMO – multiple-input multiple-output) have recently received unprecedented attention due to their extraordinary-high spectral efficiency. As any wireless communication system, they suffer from impairments of the wireless propagation channel. However, the impact of the propagation channel on MIMO system performance is much more profound and complicated than for traditional (i.e., single antenna) systems. While the MIMO system performance is much superior to that of the traditional systems, it may be significantly deteriorated by the propagation channel in some scenarios. This explains large interest in studying the MIMO propagation channel.

The use of waveguide and cavity models to study the MIMO capacity, which seems to be strange at first, has three profound reasons. First, many indoor channels can be modeled as waveguides or cavities (example: corridors, tunnels, etc.) and, hence, their capacity analysis is of practical interest. Secondly, this problem can be considered as a canonical one – its solution allows one to develop the necessary analysis techniques, which can be applied later on to many similar problems. Finally, the solution of this problems would shed some light on a MIMO structure of electromagnetic (EM) field itself, on the impact of the electromagnetism laws on the MIMO capacity in general, and, ultimately, on the relation between the laws of electromagnetism and information theory.

II. MIMO CAPACITY OF WAVEGUIDE CHANNELS

In this paper, we use the now classical Foschini-Telatar formula for the MIMO channel capacity [3-4]. For a fixed linear $m\times n$ matrix channel with additive white gaussian noise and when the transmitted signal vector is composed of statistically independent equal power components each with a gaussian distribution and the receiver knows the channel, the channel capacity is:

$$C = \log_2 \det\left(\mathbf{I} + \frac{\rho}{n}\mathbf{H}\cdot\mathbf{H}^{+}\right) \text{ bits/s/Hz,} \quad (1)$$

where n and m are the numbers of transmit and receive antennas respectively, ρ is the average signal-to-noise ratio (SNR), $\mathbf{I}$ is $m\times m$ identity matrix, $\mathbf{H}$ is the normalized channel matrix, which is considered to be frequency independent over the signal bandwidth, and "$^+$" denotes transpose conjugate. Further, we assume that m=n=N, where N is the number of waveguide modes (note that for a given frequency, N is always limited [2]). We also assume that the bandwidth is sufficiently small so that the channel is frequency flat.

The main idea for a waveguide channel is to use the eigenmodes (or simply modes) as independent sub-channels since they are orthogonal (if the waveguide is lossless and uniform) and it is well-known that the MIMO capacity is maximum for independent sub-channels. Since any field inside of the waveguide can be presented as a linear combination of the modes [2], the maximum number of independent sub-channels equals to the number of modes and there is no loss in capacity if *all* the modes are used. For lossy and/or non-uniform waveguide, there exist some coupling between the modes [2] and, hence, the

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capacity is smaller (due to the power loss as well as to the coupling). Thus, the capacity of a lossless waveguide will provide an upper bound for a true capacity since some loss and non-uniformity is always inevitable. It should be noted that if the coupling results in the sub-channel correlation less than approximately 0.5, the capacity decrease is not significant [5]. We further assume that the waveguide is lossless and is matched at both ends. In this case, the transverse electric fields for two different *E* modes, or two different *H* modes, or one *E* and one *H* mode are mutually orthogonal [2]

$$\iint_S \mathbf{E}_\mu \mathbf{E}_\nu dS = c\delta_{\mu\nu}, \qquad (2)$$

where the integral is over the waveguide cross-sectional area S, μ and ν are composite mode indices, $\delta_{\mu\nu}$ is Kronecker delta, and *c* is a constant (which depends on the power transmitted in each mode). (2) immediately suggests the system architecture to achieve the maximum MIMO capacity using the modes: at the Tx end, all the possible modes are excited using any of the well-known techniques and at the Rx end the transverse electric field is measured on the waveguide cross-sectional area (proper spatial sampling may be used to reduce the number of field sensors) and is further correlated with the distribution functions of each mode. The signals at the correlator outputs are proportional to the corresponding transmitted signals since the modes are orthogonal and, hence, there is no cross-coupling between different Tx signals. Thus, the equivalent channel matrix (i.e., Tx end-Rx end-correlator outputs) is $\mathbf{H} = \mathbf{I}_N$ (recall that the waveguide is assumed to be matched and lossless), where $\mathbf{I}_N$ is *N*x*N* identity matrix, and the capacity achieves its maximum (for given SNR ρ):

$$C = N \log_2 (1 + \rho / N) \qquad (3)$$

Knowing the number of modes N and using (3), the maximum MIMO capacity can easily be evaluated. In contrast to [1], the maximum capacity (we call it further simply "capacity") of the present MIMO architecture described above does not vary along the waveguide length and it increases with the number of modes, as one would intuitively expect. If not all the available modes are used, the capacity decreases accordingly. The capacity may also decrease if the Rx antennas measure the field at some specific points rather than the field distribution along the cross-sectional area (since the mode orthogonality cannot be efficiently used in this case). This was the case in [1] and it explains the variation of the capacity along the waveguide there. In order to evaluate the maximum capacity using (3), we further evaluate the number of modes.

III. RECTANGULAR WAVEGUIDE

Let us consider first a rectangular waveguide located along OZ axis (see Fig. 1). The field distribution at XY plane

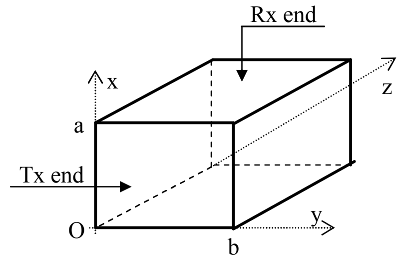

Fig. 1 Rectangular waveguide geometry.

(cross-section) for *E* and *H* modes is given by well-known expressions [2] and the variation along the OZ axis is given by $e^{-jk_z z}$, where *j* is imaginary unit, and k_z is the longitudinal component of the wavenumber:

$$k_z = \sqrt{\left(\frac{\omega}{c_0}\right)^2 - \gamma_{mn}^2}, \quad \gamma_{mn}^2 = \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2, \qquad (4)$$

where ω is the frequency, c_0 is the speed of light, and *m* and *n* designate the mode (note that *E* and *H* modes with the same (*m,n*) pair have the same γ_{mn}). The sign of k_z is chosen in such a way that the filed propagates along OZ axis (i.e., from the Tx end to the Rx end). The case of $\gamma_{mn} > \omega/c$ corresponds to the evanescent field, which decays exponentially with *z* and is negligible at few wavelength from the source [2]. Assuming that the Rx end is located is far enough from the Tx end (i.e., at least few wavelengths), we neglect the evanescent field. Hence, the maximum value of γ_{mn} is $\gamma_{mn,\max} = \omega/c$. This limits the number of modes that exist in the waveguide at given frequency ω. All the modes must satisfy the following inequality, which follows from (4):

$$\left(\frac{m}{a'}\right)^2 + \left(\frac{n}{b'}\right)^2 \le 4, \qquad (5)$$

where $a' = a/\lambda$, $b' = b/\lambda$ and λ is the free-space wavelength; and $m, n = 1, 2, ...$ for *E* mode and $m, n = 0, 1, ...,$ $\mathrm{m{+}n} \ne 0$ for *H* mode. Using a numerical procedure and (5), the number of modes *N* can be easily evaluated. A closed-from approximate expression can be obtained for large a' and b' by observing that (5) is, in fact, an equation of ellipse in terms of (*m,n*) and all the allowed (*m,n*) pairs are located within the ellipse. Hence, the number of modes is given approximately by the ratio of areas:

$$N \approx 2\frac{S_e/4}{S_0} = \frac{2\pi ab}{\lambda^2} = \frac{2\pi S_w}{\lambda^2}, \qquad (6)$$

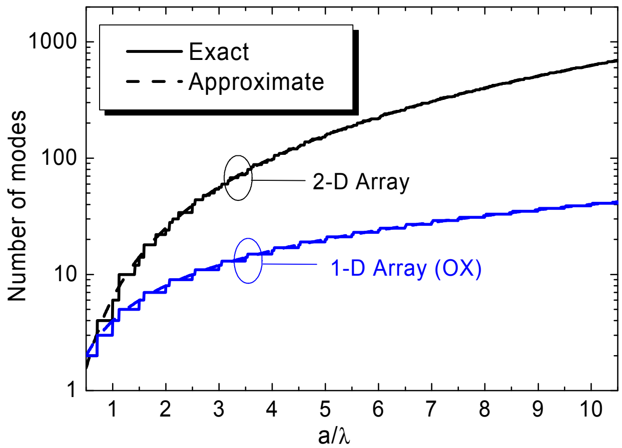

Fig. 2. Number of modes in a rectangular waveguide for *a*=*b*.

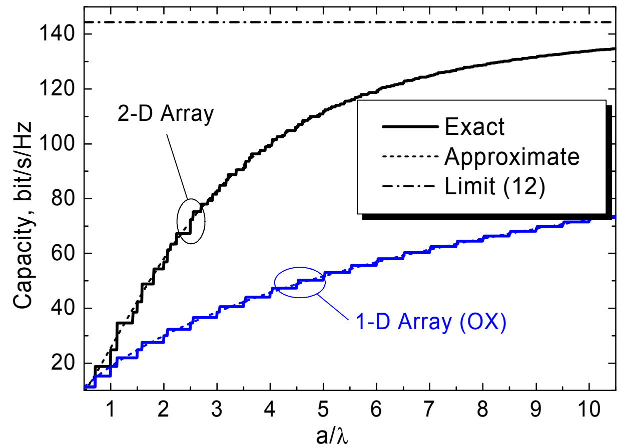

Fig. 3. MIMO capacity in a rectangular waveguide for *a*=*b* and SNR=20 dB.

where $S_e = 4\pi a'b'$ is the ellipse area, $S_0 = 1$ is the area around each (m,n) pair, $S_w = ab$ is the waveguide cross-sectional area, the factor ¼ is due to the fact that only nonnegative m and n are considered, and the factor 2 is due to the contributions of both E and H modes. As (6) demonstrates, the number of modes is determined by the ratio of the waveguide cross-section area ab to the wavelength squared. As we will see further, this is true for a circular waveguide as well. Hence, one may conjecture that this is true for a waveguide of arbitrary cross-section as well. This conjecture seems to be consistent with the spatial sampling argument (2-D sampling must be considered in this case). In fact, (6) gives the number of degrees of freedom the rectangular waveguide is able to support and which can be further used for MIMO communication. Fig. 2 compares the exact number of modes computed numerically using (5) and the approximate number (6). As one may see, (6) is quite accurate when a and b are greater then approximately a wavelength. Note that the number of modes has a step-like behavior with a/λ, which is consistent with (5). Using (3) and (6), the maximum capacity of the rectangular waveguide channel can be easily evaluated.

The analysis above assumes that the E-field (including both E_x and E_y components) is measured on the entire cross-sectional area (or at a sufficient number of points to recover it using the sampling expansion). However, it may happen in practice that only one of the components is measured, or that the field is measured only along OX (or OY) axis. Apparently, it should lead to the decrease of the available modes. This is analysed below in details.

The normalized field distribution at the waveguide cross-sectional area is [2]

$$\text{E-mode}\ \begin{cases} E_{x,mn} = \dfrac{2\pi m}{\gamma_{mn} a\sqrt{ab}} \cos\dfrac{m\pi x}{a} \sin\dfrac{n\pi y}{b} \\ E_{y,mn} = \dfrac{2\pi n}{\gamma_{mn} b\sqrt{ab}} \sin\dfrac{m\pi x}{a} \cos\dfrac{n\pi y}{b} \end{cases} \tag{7}$$

$$\text{H-mode}\ \begin{cases} E_{x,mn} = \dfrac{-2\pi n}{\gamma_{mn} b\sqrt{ab}} \cos\dfrac{m\pi x}{a} \sin\dfrac{n\pi y}{b} \\ E_{y,mn} = \dfrac{2\pi m}{\gamma_{mn} a\sqrt{ab}} \sin\dfrac{m\pi x}{a} \cos\dfrac{n\pi y}{b} \end{cases} \tag{8}$$

Let us assume that the E-field (both components) is measured along the OX axis only (this corresponds to 1-D antenna array located along OX). Due to this limitation, one can compute the correlations at the Rx using the integration over OX axis only since the field distribution along OY axis is not known. Hence, we need to find the modes that are orthogonal in the following sense:

$$I = \int_0^a \mathbf{E}_\mu \mathbf{E}_\nu dx = c\delta_{\mu\nu}\,, \tag{9}$$

Using (7), we find that two different E-modes $E_{m_1 n_1}$ and $E_{m_2 n_2}$ are orthogonal provided that $m_1 \neq m_2$; if these modes have the same m index, they are not orthogonal. The same is true about two H-modes (this can be verified using (8)) and about one E-mode and one H-mode. This results in a substantial reduction of the number of orthogonal modes since, in the general case, two E-modes are orthogonal if at least one of the indices is different, i.e. if $m_1 \neq m_2$ <u>or</u> $n_1 \neq n_2$. Surprisingly, if one measures only E_x component in this case, the modes are still orthogonal provided that $m_1 \neq m_2$. Hence, if the receive antenna array is located along OX axis, there is no need to measured E_y component – it does not provide any additional degrees of freedom, which can be used for MIMO communications (recall that only orthogonal modes can be used). The number of orthogonal modes can be evaluated using (5):

$$N_x \approx \frac{4a}{\lambda}, \qquad (10)$$

This corresponds to $2a/\lambda$ degrees of freedom for each (E and H) field. Note that this result is similar to that obtained using the spatial sampling argument, i.e., independent field samples (which are, in fact, the degrees of freedom) are located at $\lambda/2$.

The similar argument holds true when the receive array is located along OY axis. In this case two modes are orthogonal provided that $n_1 \neq n_2$ and there is also no need to measure the E_x component. The number of orthogonal modes is approximately

$$N_y \approx \frac{4b}{\lambda}, \qquad (11)$$

Fig. 3 shows the MIMO capacity of a rectangular waveguide (the same as in Fig. 1) for SNR $\rho = 20$ dB. Note that the capacity saturates as a/λ increases. This is because (3) saturates as well as N increases:

$$\lim_{N\to\infty} C = \frac{\rho}{\ln 2} \qquad (12)$$

C in (3) can be expanded as

$$C = \frac{\rho}{\ln 2}\sum_{i=0}^{\infty}\frac{(-1)^i}{i+1}\left(\frac{\rho}{N}\right)^i \qquad (13)$$

For large N, i.e. for small ρ/N, this series converges very fast and it can be approximated by first two terms:

$$C \approx \frac{\rho}{\ln 2}\left(1 - \frac{\rho}{2N}\right) \qquad (14)$$

The capacity does not change substantially when the contribution of the 2nd term is small:

$$\frac{\rho}{2N} << 1 \Rightarrow N > N_{\max} \approx \rho \qquad (15)$$

N_{max} is the maximum "reasonable" number of antennas (modes) for given SNR (or vice versa): if N increases above this number, the capacity does not increase significantly. It may be considered as a practical limit (since further increase in capacity is very small and it requires for very large increase in complexity). Using (6) and (10), one finds the maximum "reasonable" size of the waveguide for the case of 2-D and 1-D arrays correspondingly:

$$\frac{a_{\max}}{\lambda} \approx \sqrt{\frac{\rho}{2\pi}} \text{ (2-D array)}, \quad \frac{a_{\max}}{\lambda} \approx \frac{\rho}{4} \text{ (1-D OX array)}, \qquad (16)$$

Note that Fig. 2 shows, in fact, the fundamental limit of the waveguide capacity, which is imposed jointly by the laws of information theory and electromagnetism.

IV. RECTANGULAR CAVITY

The analysis of MIMO capacity in cavities is very different from that in waveguides in one important aspect. Namely, the modes of a cavity exist only for some finite discrete set of frequencies (recall that, as in the case of waveguide, we consider a lossless cavity). Hence, there may be no modes for an arbitrary frequency. To avoid this problem, we evaluate the number of modes for a given bandwidth, $f \in [f_0, f_0 + \Delta f]$, starting at f_0. For a rectangular cavity, the wave vector must satisfy [2]:

$$k^2 = \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2 + \left(\frac{\pi p}{c}\right)^2 = \left(\frac{\omega}{c_0}\right)^2, \qquad (17)$$

where c is the waveguide length (along OZ axis in Fig. 1), and p is a non-negative integer; $m,n = 1,2,3,...,\ p = 0,1,2,...$ for E-modes, and $m,n = 0,2,3,...,\ p = 1,2,...$ for H-modes ($m = n = 0$ is not allowed). Noting that (17) is a equation of a sphere in terms of (m,n,p), the number of modes with $k \in [k_0, k_0 + \Delta k]$ can be found as the number of (m,n,p) points between two spheres with radiuses of k_0 and $k_0 + \Delta k$ correspondingly. Fig. 4 gives a 2-D illustration of this procedure. Using the ratio of areas approach described above, the number of modes is approximately:

$$N_c \approx 2\frac{V_e/8}{V_0} = \frac{8\pi V_c}{\lambda^3}\frac{\Delta f}{f_0}, \qquad (18)$$

where $V_e = 4\pi k^2 \Delta k$ is the volume between the two spheres, $V_0 = \pi^3/V_c$ is the volume around each (m,n,p) point, $V_c = abc$ is the cavity volume; factor 2 is due to two types of modes, and factor 1/8 is due to the fact that only

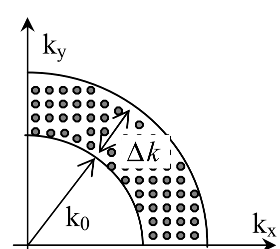

Fig. 4. 2-D illustration of wavenumber space filling.

nonnegative values of (m,n,p) are allowed. An important conclusion from (18) is that the number of modes is determined by the cavity volume expressed in terms of wavelength and by the normalized bandwidth. Detailed analysis shows that (18) is accurate for large a, b, and c, and if $c/\lambda < f_0/4\Delta f$ (see eq. 20 below and related discussion).

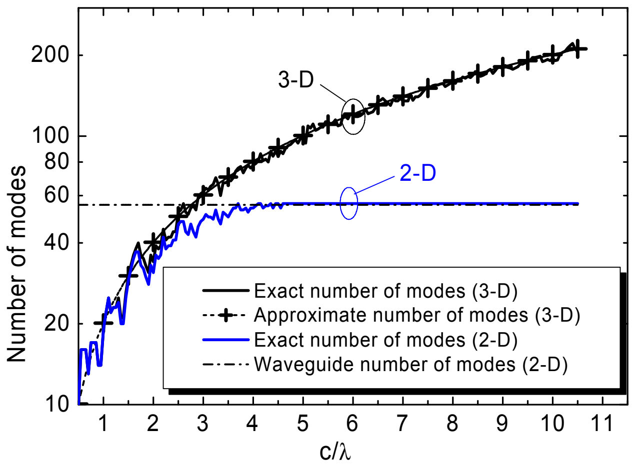

Fig. 5. Number of orthogonal modes in a rectangular cavity for $a = 4\lambda,\ b = 2\lambda$ **and** $\Delta f / f_0 = 0.01$**.**

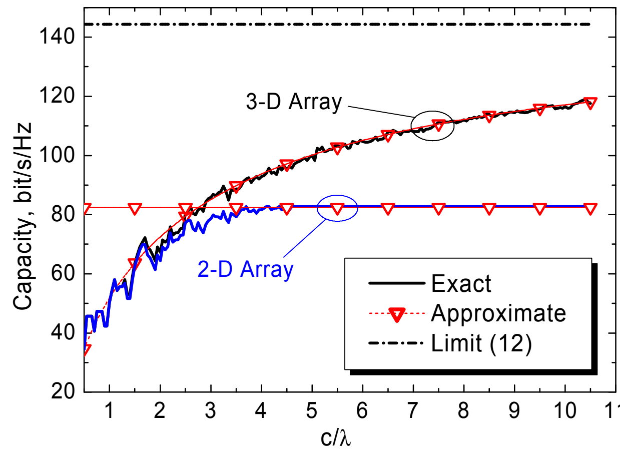

Fig. 6. Capacity in a rectangular cavity for $a = 4\lambda,\ b = 2\lambda$ **and** $\Delta f / f_0 = 0.01$**.**

It should be noted that the mode orthogonality for cavities is expressed through the volume integral (over the entire waveguide volume),

$$\iiint_{V_c} \mathbf{E}_\mu \mathbf{E}_\nu dV = c\delta_{\mu\nu}, \qquad (19)$$

and, hence, all the modes are orthogonal provided that the field is measured along all 3 dimensions, which, in turn, means that a 3-D arrays must be used, which may not be feasible in practice. If only 2-D arrays are used, then the mode orthogonality is expressed as for a waveguide, i.e. (2), and, consequently, only those modes are orthogonal that have different (m,n) indices. The use of a 2-D array results in significant reductions of the number of modes for large c, as Fig. 5 demonstrates. Note that for small c, there is no loss in the number of orthogonal modes. This is because different p correspond in this case to different (m,n) pairs (this can also be seen from (17)). However, as c increases, different p may include the same (m,n) pairs, which results in the number loss if a 2-D array is used. In fact, the 2-D case with large c is the same as the waveguide case (with the same cross-sectional area), as it should be. The value of *c* for which the cavity has the same number of orthogonal modes as the corresponding waveguide can be found from the following equation:

$$N_c \approx N_w \Rightarrow \frac{c_t}{\lambda} = \frac{f_0}{4\Delta f}, \qquad (20)$$

Hence, if 2-D antenna arrays are used and $c \geq c_t$, the waveguide model provides approximately the same results as the cavity model does, i.e. the cross-section has the major impact on the capacity, while the effect of cavity length is negligible. The waveguide model should be used to evaluate the number of orthogonal modes (and capacity) in this case because it is more simple to deal with. For example, a long corridor can be modelled as a waveguide rather than cavity (despite of the fact that it is closed and looks like a cavity). Fig. 6 shows the capacity in the cavity. While the capacity of a 2-D array system saturates like the waveguide capacity, which is limited by a and b, the capacity of a 3-D system is larger and saturates at the value given by (12). It should be noted that (12) is the capacity limit due to the information theory laws, and (6), (10), (11), and (18) are the capacity limits due to the laws of electromagnetism (i.e., limited due to the number of degrees of freedom of the EM field).

V. Circular Waveguide

A circular waveguide can be analyzed in a similar way. In this case,

$$\gamma_{mn} = \frac{p_{mn}}{a} \text{ (E mode)}, \gamma_{mn} = \frac{p'_{mn}}{a} \text{ (H mode)}, \qquad (21)$$

where a is the waveguide radius, p_{mn} is the n-th root of $J_m(x) = 0$, and, p'_{mn} is the n-th root of $\partial J_m(x)/\partial x = 0$, $J_m(x)$ is m-th order Bessel function of the first kind [2]. Hence, the number of modes can be found from the following:

$$p_{mn} \leq 2\pi a' \text{ (E modes)},\ p'_{mn} \leq 2\pi a' \text{ (H modes)}, \qquad (22)$$

where $a' = a/\lambda$. A closed-form approximate expression for large a' can be obtained using the large-argument approximation of the Bessel function:

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{m\pi}{2}\right), \qquad (23)$$

(22) can be approximated as

$$n + \frac{m}{2} \leq 2a' - \frac{3}{4} \text{ (E modes)},\ n + \frac{m}{2} \leq 2a' - \frac{1}{4} \text{ (H modes)}, \qquad (24)$$

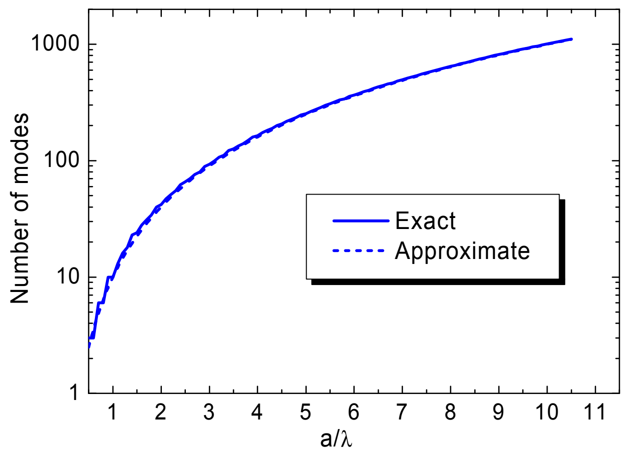

Fig. 7. Number of modes in a circular waveguide.

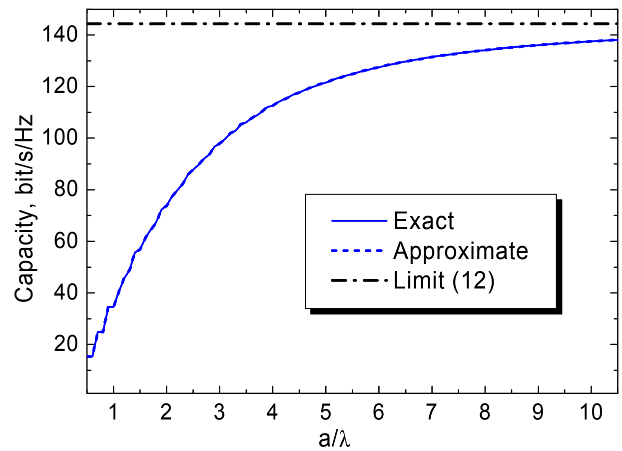

Fig. 8. MIMO capacity in a circular waveguide for SNR=20 dB.

Using the same ratio of areas approach, the number of modes is approximately

$$N \approx \frac{10a^2}{\lambda^2}, \tag{25}$$

As it can be seen from Fig. 7, this approximation is quite accurate for $a/\lambda \geq 1$. It is interesting to note that, in both cases (i.e., rectangular and circular waveguides), the number of modes is determined by the waveguide cross-sectional area expressed in terms of the wavelength in a way similar to an aperture antenna gain. One may speculate that this is true in the case of an arbitrary cross-section as well.

VI. Conclusion

MIMO capacity of waveguide and cavity channels has been discussed in this paper. There are three profound reasons for this: (i) waveguide / cavity models can be used to model corridors, tunnels and other confined space channels, (ii) this is a canonical problem; its analysis allows to develop appropriate techniques, which can be further used for more complex problems, (iii) it allows to shed light on the relation between information theory and electromagnetism and, in particular, to establish the limits imposed by the laws of electromagnetism on achievable channel capacity.

We have demonstrated that the number of degrees of freedom (i.e., the number of orthogonal modes) of the electromagnetic field inside of a rectangular or circular waveguide is determined by the waveguide cross-sectional area expressed in terms of the wavelength. All these degrees of freedom can be used for MIMO communications. A similar approach can be applied to cavity channels as well.

Overall, the approach presented in this paper demonstrates that the information theory and electromagnetism techniques can be used together to get a new insight into the performance of such well-known structures as, for example, waveguides (note that it can be applied to optical waveguides as well, where the capacity is of large importance).

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