ELG5375: Digital Communications

Lecture 9

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M-ary Modulation Schemes

- $\log_2 M$ bits are transmitted by each symbol
- M-PSK: the phase can take M different values

$$x_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_c t + \frac{2\pi}{M} i\right), \quad i = 0, 1, \ldots, M-1, \qquad (1)$$

$$0 \le t \le T$$

- symbol error probability (SER):

$$P_{se} \approx 2Q\left(\sqrt{\frac{2E}{N_0}} \sin\left(\frac{\pi}{M}\right)\right) = \alpha Q\left(\sqrt{\frac{\beta E}{N_0}}\right) \qquad (2)$$

- **Q:** constellation example? Minimum distance $d_{\min}$?

M-ary PSK: bandwidth/energy/power efficiency

Table 6.4 Bandwidth and Power Efficiency of M-ary PSK Signals

M	2	4	8	16	32	64
$\eta_B = R_b/B$ *	0.5	1	1.5	2	2.5	3
E_b/N_o for BER=10^{-6}	10.5	10.5	14	18.5	23.4	28.5

* B: First null bandwidth of M-ary PSK signals

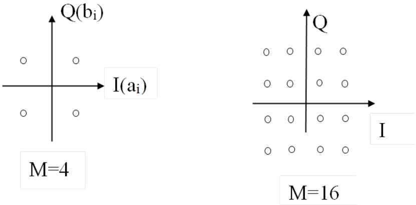
T.S. Rappaport, *Wireless Communications*, Prentice Hall, 2002

- bandwidth for $p(t)$ = rectangular pulse or RC pulse with $\alpha = 1$
- reminder: $E_b = E_s / \log_2 M$

M-QAM

$$x_i(t) = a_i \cos \omega_c t - b_i \sin \omega_c t \;\Leftrightarrow\; A_i = a_i + jb_i,\; x_i(t) = \mathrm{Re}\{A_i e^{j\omega_c t}\} \tag{3}$$

a_i and b_i = I/Q data



The BER of M-QAM ($M = 2^k$ and k is even):

$$P_{eb} \approx \frac{4\left(1 - \frac{1}{\sqrt{M}}\right)}{\log_2 M} Q\left[\sqrt{\frac{3\log_2 M}{M-1} \cdot \frac{E_b}{N_0}}\right] \tag{4}$$

M-QAM: bandwidth/energy/power efficiency

Table 6.5 Bandwidth and Power Efficiency of QAM [Zie92]

M	4	16	64	256	1024	4096
η_B	1	2	3	4	5	6
E_b/N_o for BER $= 10^{-6}$	10.5	15	18.5	24	28	33.5

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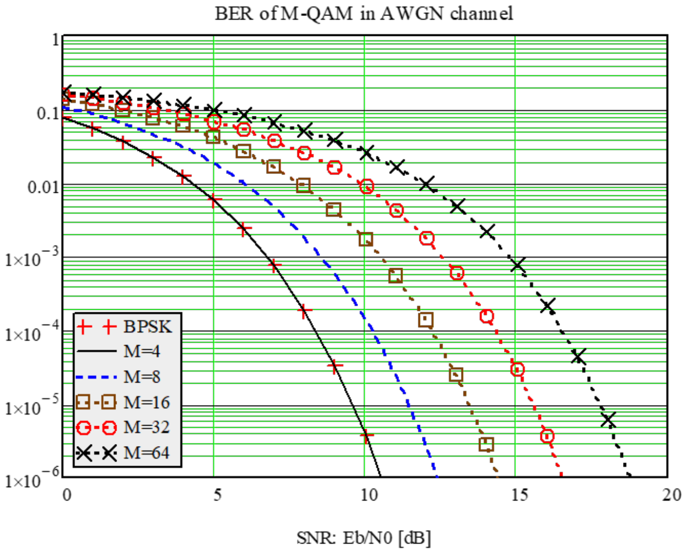
- The threshold SNR: $\gamma_{\text{th}} \sim 10 \log M$
- **Q:** which is better, M-QAM or M-PSK?

M-QAM BER in AWGN



M-QAM vs. M-PSK: which is better?

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M-ary frequency Shift Keying (FSK)

- M-ary FSK

$$x_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \varphi), \quad i = 1, \ldots, M, \quad 0 \leq t \leq T \tag{5}$$

- signal orthogonality imposes a limit on $\Delta\omega = \omega_{i+1} - \omega_i$.
- **Q:** find min $\Delta\omega$ such that the signals $x_i(t)$ and $x_{i+1}(t)$ are orthogonal

- the BER/SER of orthogonal BFSK (coherently detected):

$$P_e = Q(\sqrt{\gamma}) \tag{6}$$

M-ary frequency Shift Keying (M-FSK)

- The tight upper bound (holds with $\approx$) on the SER is

$$P_{es} \leq (M-1)\, Q(\sqrt{\gamma}) \qquad (7)$$

for orthogonal signals and coherent demodulation.

Table 6.6 Bandwidth and Power Efficiency of Coherent M-ary FSK [Zie92]

M	2	4	8	16	32	64
η_B	0.4	0.57	0.55	0.42	0.29	0.18
E_b/N_o for BER = 10^{-6}	13.5	10.8	9.3	8.2	7.5	6.9

T.S. Rappaport, *Wireless Communications*, Prentice Hall, 2002

- which is better, M-QAM, M-PSK or M-FSK?

M-FSK vs. M-PSK vs. M-QAM: which is better?

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Analog vs. digital SNR

SNR in analog (RF) systems is

$$SNR_a = P_s / P_n \tag{8}$$

P_s, P_n are the signal and noise powers. Note that

$$P_s = \frac{E_s}{T_s},\ R_s = \frac{1}{T_s},\ P_n = N_0 \cdot \Delta f \tag{9}$$

so that

$$\gamma = \frac{E_s}{N_0} = \frac{P_s \cdot T_s}{P_n/\Delta f} = \frac{P_s}{P_n} \cdot \frac{\Delta f}{R_s} = SNR_a \cdot \frac{\Delta f}{R_s} \tag{10}$$

i.e. $\gamma = SNR_a$ if $\Delta f = R_s$

Analog vs. digital SNR

- For $R_s = \Delta f$, they are the same.
- $\Delta f / R_s$ is inverse of the bandwidth efficiency $R_s/\Delta f$

- Similarly, for per-bit SNR,

$$\gamma_b = \frac{E_b}{N_0} = SNR_a \cdot \frac{\Delta f}{R} = \frac{SNR_a}{\eta} \tag{11}$$

where R = bit rate (bit/s), $\eta = R/\Delta f$ = SE

SER and Signal Constellation

- SER/BER: can be difficult to compute in general
- various bounds are useful
- for **any** constellation, the SER satisfies (UB)

$$P_e \leq (M-1)\, Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) \tag{12}$$

$d_{\min}$ is the minimum distance in the constellation

- i.e. SER decreases exponentially fast with SNR, **for any constellation geometry**
- used extensively in communication/information theory
- **Q:** prove the UB

SER and Signal Constellation

- $d_{\min}$, $N_{min} = ?$

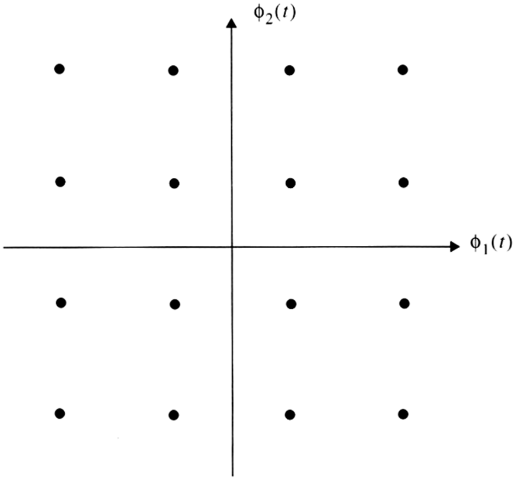

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Figure 6.47 Constellation diagram of an M-ary QAM ($M = 16$) signal set.

SER/BER at high SNR

- SER/BER: difficult to compute in general
- much simpler at high SNR, via approximations
- a high-SNR approximation for SER:

$$P_{es} \approx N_{min}\, Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) \tag{13}$$

  N_{min} = the number of nearest neighbours
- BER at high SNR

$$P_{eb} \approx \frac{1}{\log_2 M}\, P_{es} \tag{14}$$

- **Q:** how can you explain (13), (14)?

Summary

- M-ary modulation formats. Comparisson.
- Power and bandwidth efficiency.
- BER and SER.
- High-SNR approximations
- The role of $d_{min}, \ N_{min}$

Reading

- S. Haykin, Digital Communication Systems, Wiley, 2014.
- J.M. Wozencraft, I.M. Jacobs, Principles of Communication Engineering, Wiley, 1965.
- R.E. Ziemer, W.H. Tranter, Principles of Communications, Wiley, New York, 2009.
- Other books (see the reference list).

Note: Do not forget to do end-of-chapter problems. Remember the learning efficiency pyramid!