ELG5375: Digital Communications

Lecture 8

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Carrier modulation: DSB-AM

- shift the baseband signal $s(t)$ to carrier frequency f_c:

$$x(t) = s(t) \cos \omega_c t \tag{1}$$

- double-sideband AM (DSB-AM)
- double-sided spectrum (FT):

$$S_x(f) = \frac{1}{2}(S_s(f - f_c) + S_s(f + f_c)) \tag{2}$$

- also applies to ESD/PSD (under mild condition)
- doubles the bandwidth: $\Delta f_x = 2\Delta f_s$

- **Q:** how to recover $s(t)$ back from $x(t)$?
- **Q:** how to recover SE back?

Carrier modulation: BPSK

- binary phase shift keying (BPSK)
- sending 1 bit per symbol
- $s(t) = m = \pm 1$ (2-PAM):

$$x(t) = m\,A_c \cos\omega_c t = A_c \cos(\omega_c t + \theta_m),\ \ \theta_m = 0, \pi \tag{3}$$

- this corresponds to $p(t) = A_c \Pi(t/T)$
- can use different $p(t)$:

$$x(t) = m\,p(t)\cos\omega_c t = p(t)\cos(\omega_c t + \theta_m) \tag{4}$$

- bandwidth $\Delta f_x = 2\Delta f_p$
- **Q:** find ESD of $x(t)$ assuming ESD of $p(t)$ is known
- **Q:** optimal Rx for this signal in AWGN?

Carrier modulation: BPSK

- and sequential transmission of many bits:

$$x(t) = \sum_i m_i p(t/T - i) \cos \omega_c t \qquad (5)$$
$$= \sum_i p(t/T - i) \cos(\omega_c t + \theta_i)$$

- $x(t) =$ RF pulse train
- **Q:** optimal Rx for this signal in AWGN?
- **Q:** ESD/PSD = ?

Carrier modulation: QAM

- quadrature amplitude modulation (QAM)
- use both cos (I) and sin (Q) carriers:

$$x(t) = s_I(t) \cos \omega_c t - s_Q(t) \sin \omega_c t \tag{6}$$

- doubles the number of messages (bits) send
- same bandwidth as for DSB-AM: $\Delta f_x = 2\Delta f_s$
- i.e. **doubles SE**
- **Q:** how to recover (losslessly) $s_{I/Q}(t)$ back from $x(t)$?
- **Q:** spectrum, ESD/PSD of $x(t)$?

Carrier modulation: QPSK

- quadrature phase shift keying (QPSK)
- sending 2 bits per symbol (in the same bandwidth)
- $s_I(t) = m_I = \pm 1$, $s_Q(t) = m_Q = \pm 1$ (2-PAM):

$$x(t) = m_I A_c \cos \omega_c t - m_Q A_c \sin \omega_c t \quad (7)$$

$$= \sqrt{2} A_c \cos(\omega_c t + \theta_m),\ \theta_m = \pm\pi/4,\ \pm 3\pi/4 \quad (8)$$

- this corresponds to $p(t) = A_c \Pi(t/T)$
- can use different $p(t)$: $x(t) = p(t) \cos(\omega_c t + \theta_m)$
- $\Delta f_x = 2\Delta f_p$ (no increase w.r.t. BPSK)

- **Q:** prove (8), find relationship between θ_m and m_I, m_Q
- **Q:** optimal Rx for this signal in AWGN?

Signal Constellation

- key: geometric representation of signals, via signal space
- signals as vectors on complex plane: in-phase (I)/quadrature (Q) parts
- carrier-modulated signal

$$\begin{aligned} x(t) &= A(t)\cos(\omega t + \varphi(t)) \\ &= A(t)\cos\varphi(t)\,\cos\omega t - A(t)\sin\varphi(t)\,\sin\omega t \\ &= \underbrace{A_I(t)\cos\omega t}_{I} - \underbrace{A_Q(t)\sin\omega t}_{Q} \end{aligned} \tag{9}$$

$$A_I(t) = A(t)\cos\varphi(t),\; A_Q(t) = A(t)\sin\varphi(t)$$

where A_I, A_Q are I/Q amplitudes

- complex amplitude (envelope) $A_c = A_I + jA_Q$

Complex-valued form

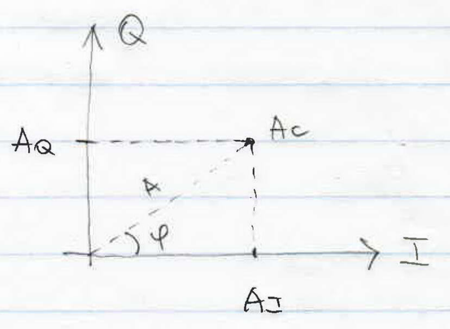
- via complex envelope/amplitude A_c:

$$x(t) = A(t)\cos(\omega t + \varphi(t)) = \mathrm{Re}\Big\{A(t)e^{j(\omega t+\varphi(t))}\Big\}$$
$$= \mathrm{Re}\{A_c(t)e^{j\omega t}\} \quad (10)$$

  - $A_c = Ae^{j\varphi} =$ complex amplitude
  - $\omega =$ carrier frequency
  - $A =$ carrier amplitude/envelope (real)

$$A = |A_c| = \sqrt{A_I^2 + A_Q^2},\ \varphi = \arg\{A_c\} \overset{?}{=} \tan^{-1}\frac{A_Q}{A_I} \quad (11)$$

Signal constellation on complex plane

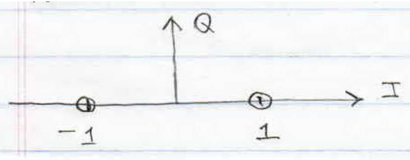


$$\left.\begin{aligned} A &= A(t) \\ \varphi &= \varphi(t) \\ A_c &= A_c(t) \end{aligned}\right\} \leftarrow \text{modulation}$$

BPSK constellation

$$x(t) = s(t)\cos\omega t;$$

$$s(t) = m = \pm 1, \quad A_I = \pm 1, \quad A_Q = 0, \quad A_c = \pm 1, \quad \varphi = 0, 180^\circ$$



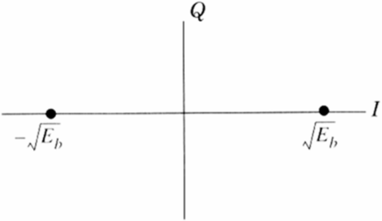


Figure 6.21 BPSK constellation diagram. T.S.

Rappaport, *Wireless Communications*, Prentice Hall, 2002

QPSK constellation

- BPSK: 1 bit/symbol; $\varphi = 0$ or π
- QPSK: 2 bits/symbol; φ: 4 values
- 2 forms:

$$x_i(t) = A\cos(\omega t + \varphi_i)$$
$$\text{(a) } \varphi_i = i\pi/2,\ \ i = 0, 1, 2, 3 \qquad (12)$$
$$\text{(b) } \varphi_i = i\pi/2 + \pi/4$$

QPSK constellation

- (b) combination of I and Q BPSK:

$$x(t) = a_i \cos \omega t - b_i \sin \omega t \tag{13}$$

- $a_i, b_i = \pm 1 \;\rightarrow$ I and Q data

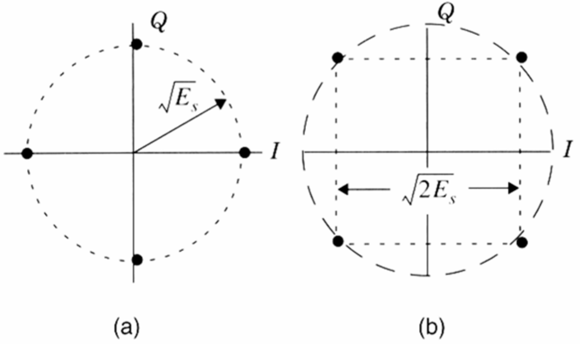


Figure 6.26 (a) QPSK constellation

T.S. Rappaport, *Wireless Communications*, Prentice Hall, 2002

QPSK: Properties

- BPSK: 1 bit/symbol
- QPSK: 2 bit/symbol $\rightarrow$ twice SE (same Δf)
- spectral efficiency (SE), with $p(t) = \text{sinc}(t/T_s)$:

$$\text{QPSK:} \quad \text{SE} = \eta = \frac{R_b}{\Delta f} = \frac{2\,\text{bit}/T_s}{1/T_s} = 2 \tag{14}$$

$$\text{BPSK:} \quad \eta = \frac{1\,\text{bit}/T_s}{1/T_s} = 1 \tag{15}$$

- I/Q data sequences: constructed in the same way as for BPSK,

$$m_I(t) = \sum_i a_i\, p(t - iT),\; m_Q(t) = \sum_i b_i\, p(t - iT) \tag{16}$$

i.e. separate BPSKs over I and Q channels.

QPSK pulse train

- Bandpass modulated signal (RF pulse train):

$$x(t) = A \sum_i p(t - iT) \cos(\omega t + \theta_i) \tag{17}$$

$\theta_i =$ encodes data, e.g.

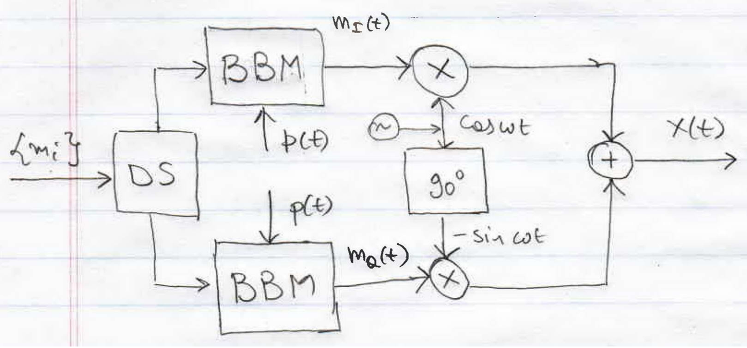
$$00 \rightarrow \theta_1, \quad 01 \rightarrow \theta_2, \quad 10 \rightarrow \theta_3, \quad 11 \rightarrow \theta_4$$

- Gray bit mapping: adjacent points differ by 1 bit

$$00 \rightarrow \pi/4, \quad 01 \rightarrow 3\pi/4, \quad 11 \rightarrow -3\pi/4, \quad 10 \rightarrow -\pi/4$$

QPSK Modulator (Tx)

QPSK $= 2\times$ BPSK



QPSK Modulator (Tx)

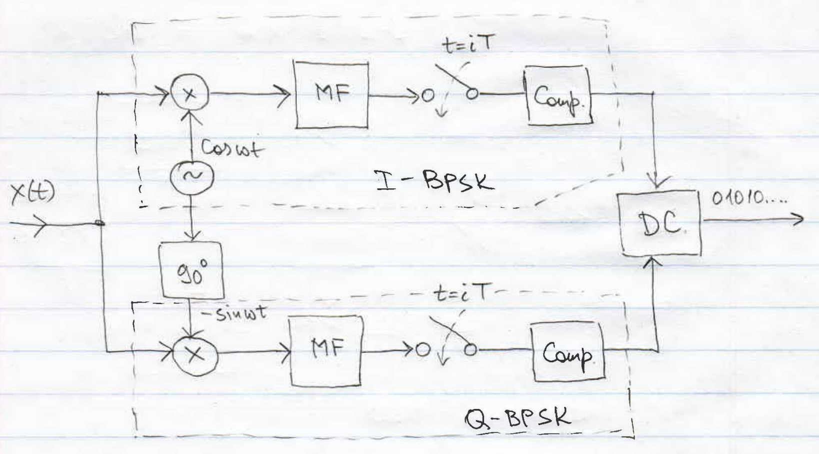
- BBM = baseband binary (BPSK) modulator
- DS = data splitter

$$m_I(t) = \sum_i a_i \, p(t - iT), \quad m_Q(t) = \sum_i b_i \, p(t - iT) \tag{18}$$

$$m_i \rightarrow \{a_i, b_i\}, \qquad a_i, b_i = \pm 1$$

- $m_I(t), m_Q(t)$ = baseband I/Q signals.

QPSK Demodulator (Rx): 2 × BPSK



- MF = baseband matched filter (to $p(t)$),
- DC = data combiner.

QPSK BER

Probability of bit error (bit error rate, BER):

$$P_{eb} = Q(\sqrt{\gamma}) = Q\left(\sqrt{2\gamma_b}\right) \tag{19}$$

γ_b is the per-bit SNR,

$$\gamma_b = \frac{E_b}{N_0} = \frac{E_s}{2N_0} = \frac{\gamma}{2} \tag{20}$$

γ is the per-symbol SNR,
$E_s = 2E_b$ is the symbol energy,
E_b is the energy per bit

Key parameters of QPSK

- bandwidth of QPSK with $p(t) =$ RC pulse
- baseband Δf_p and RF $\Delta f_{RF} = 2\Delta f_p$:

$$\Delta f_p = \frac{1+\alpha}{2}\frac{R_b}{2}, \quad \Delta f_{RF} = \frac{1+\alpha}{2}R_b \tag{21}$$

- bit/symbol rate:

$$R_b = 2R_s = 2/T_s = \text{bit rate [bits/s]} \tag{22}$$

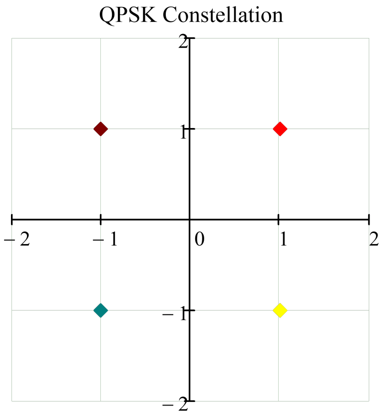
- bit interval $T_b = T_s/2$
- spectral efficiency (SE):

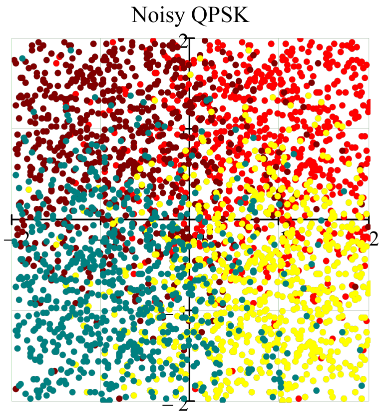
$$\eta = \frac{2}{1+\alpha} \quad \text{[bits/s/Hz]} \Rightarrow \text{QPSK} = 2 \times \text{BPSK} \tag{23}$$

Key parameters of any digital modulation

- data rate R_b
- BER (bit error probability) P_e
- bandwidth: baseband Δf_p or RF Δf_{RF}
- spectral efficiency (SE) η

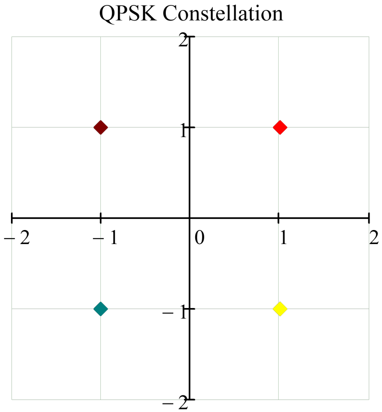
QPSK Constellation in Noise: SNR = 0 dB

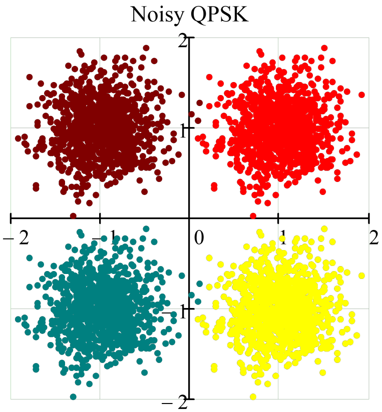




$N = 1000$ symbols transmitted.

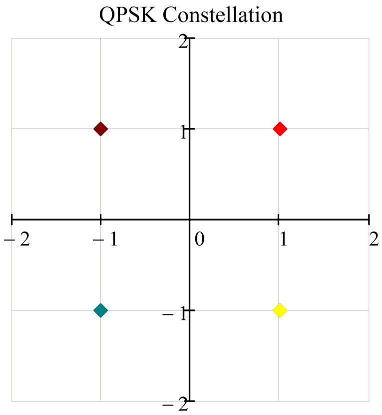
QPSK Constellation in Noise: SNR = 10 dB

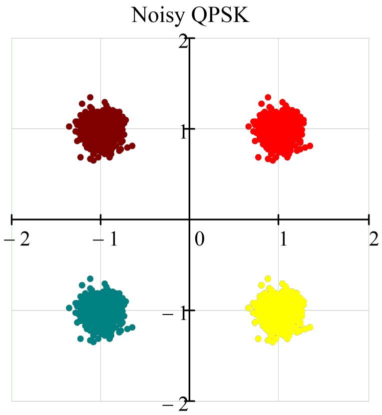




$N = 1000$ symbols transmitted.

QPSK Constellation in Noise: SNR = 20 dB





$N = 1000$ symbols transmitted.

Quadrature Amplitude Modulation (QAM)

- very popular – used by most modern digital systems
- efficient ("capacity-achieving")
- key idea: I and Q channels simultaneously with M-PAM
- 2 $\times$ rate of 1 channel (I or Q)
- independent I/Q channels as

$$\int_T \cos\omega t \, \sin\omega t \, dt = 0 \tag{24}$$

- combined with baseband M-PAM:

$$s_i(t) = A_i p(t), \qquad i = 1, \ldots, M \tag{25}$$

M-ary QAM

$$M\text{-QAM} = \underbrace{\sqrt{M}\text{-PAM}}_{I} \times \underbrace{\sqrt{M}\text{-PAM}}_{Q}$$

RF (modulated) signal of M-QAM: use $\sqrt{M}$-PAM on I and Q:

$$x(t) = m_I(t) A_c \cos\omega_c t - m_Q(t) A_c \sin\omega_c t$$
$$m_I(t) = a\, p(t), \quad m_Q(t) = b\, p(t), \quad 0 \le t \le T \tag{26}$$

- a, b represent I and Q bits of $\sqrt{M}$-PAM

M-PAM (reminder)

allowed amplitude levels for M-PAM:

$$
\begin{aligned}
& a,\ b \in [a_1, a_2 ... a_M], \\
& a_m = (2m - M - 1),\ m = 1...M
\end{aligned}
\tag{27}
$$

e.g. for $M = 4$:

$$
a,\ b \in [-3, -1, 1, 3] \tag{28}
$$

Signal space of QAM

- basis functions:

$$\phi_1(t) = \sqrt{\frac{2}{E_p}} p(t) \cos \omega_c t, \quad \phi_2(t) = \sqrt{\frac{2}{E_p}} p(t) \sin \omega_c t \qquad (29)$$

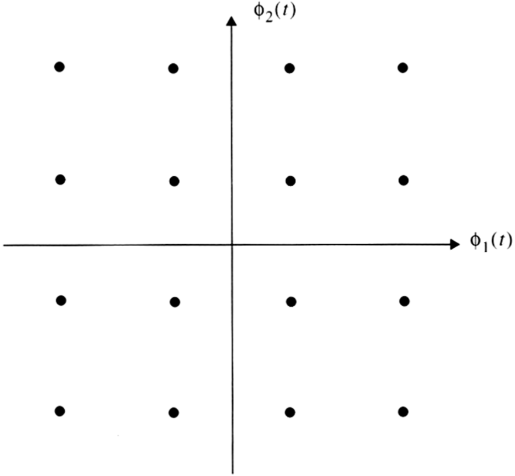
- orthogonality property:

$$\int_T \phi_1(t)\phi_2(t)\, dt = 0 \qquad (30)$$

if $S_{p^2}(f)\, S_{\sin 2\omega_c t}(f) = 0 \ \forall \ f$,
i.e. the spectra of $p^2(t)$ and $\sin 2\omega_c t$ do not overlap.

- **Q:** prove this

Constellation of M-QAM



T.S. Rappaport, Wireless Communications, Prentice Hall, 2002

Figure 6.47 Constellation diagram of an M-ary QAM ($M = 16$) signal set.

M-QAM: symbol error rate (SER)

- minimum symbol energy $E_{\min}$:

$$E_{\min} = \frac{A_c^2}{2} E_p,\ \ E_p = \int_T p^2(t)\, dt \tag{31}$$

- probability of symbol error = symbol error rate (SER) P_s:

$$\begin{aligned} P_s &\approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{2E_{\min}}{N_0}}\right) \\ &= 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3E_{av}}{(M-1)N_0}}\right) \end{aligned} \tag{32}$$

where E_{av} = average symbol energy

$$E_{av} = 2(M-1)E_{\min}/3 \tag{33}$$

M-ary modulation: BER vs. SER

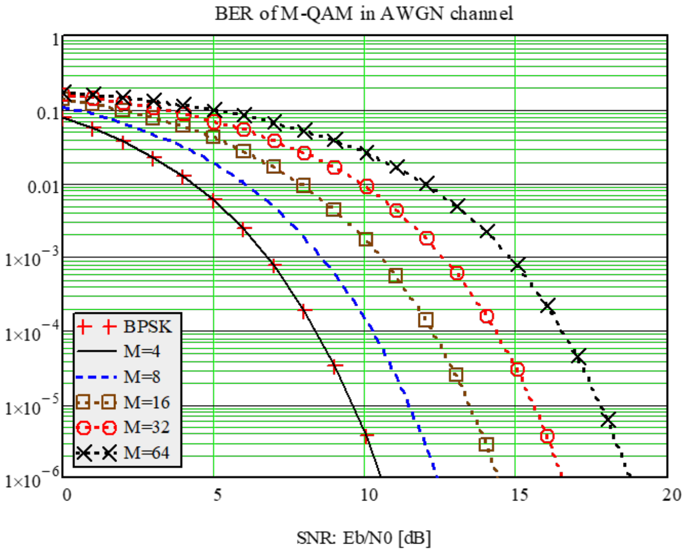
The BER P_b for general M-ary modulation satisfies

$$\frac{1}{\log_2 M} P_s \leq P_b \leq P_s, \;\; P_b \approx \frac{1}{\log_2 M} P_s \tag{34}$$

and the approximation holds for small P_s.

Q: prove this

M-QAM BER in AWGN



M-QAM BER in AWGN

- Q.: reproduce the graph
- Q.: how much extra SNR do you need to add 1 extra bit at the same BER?
- adaptive modulation: keep BER (almost) constant.

IEEE 802.11n WiFi standard

MCS Index	Type	Coding Rate	Spatial Streams	Data Rate (Mbps) with 20 MHz CH		Data Rate (Mbps) with 40 MHz CH	
				800 ns	400 ns (SGI)	800 ns	400 ns (SGI)
0	BPSK	1 / 2	1	6.50	7.20	13.50	15.00
1	QPSK	1 / 2	1	13.00	14.40	27.00	30.00
2	QPSK	3 / 4	1	19.50	21.70	40.50	45.00
3	16-QAM	1 / 2	1	26.00	28.90	54.00	60.00
4	16-QAM	3 / 4	1	39.00	43.30	81.00	90.00
5	64-QAM	2 / 3	1	52.00	57.80	108.00	120.00
6	64-QAM	3 / 4	1	58.50	65.00	121.50	135.00
7	64-QAM	5 / 6	1	65.00	72.20	135.00	150.00
8	BPSK	1 / 2	2	13.00	14.40	27.00	30.00
9	QPSK	1 / 2	2	26.00	28.90	54.00	60.00
10	QPSK	3 / 4	2	39.00	43.30	81.00	90.00
11	16-QAM	1 / 2	2	52.00	57.80	108.00	120.00
12	16-QAM	3 / 4	2	78.00	86.70	162.00	180.00
13	64-QAM	2 / 3	2	104.00	115.60	216.00	240.00
14	64-QAM	3 / 4	2	117.00	130.00	243.00	270.00
15	64-QAM	5 / 6	2	130.00	144.40	270.00	300.00
16	BPSK	1 / 2	3	19.50	21.70	40.50	45.00
...	...	...	...	...	...	...	...
31	64-QAM	5 / 6	4	260.00	288.90	540.00	600.00

802.11n Primer, Whitepaper, AirMagnet, August 05, 2008.

4G/LTE/5G systems

- Optimized for high-speed data service (Internet), VoIP.
- Modulation: OFDM + QPSK/16QAM/64QAM, up to 20MHz bandwidth.
- 5G: improves 4G, supports most 4G configurations + much more.

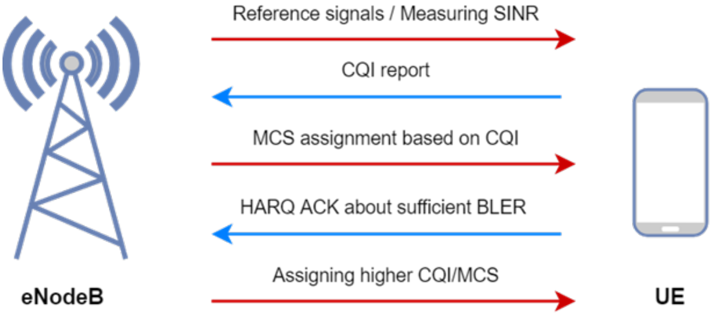
4/5G Adaptive Modulation and Coding

SINR and CQI mapping table in LTE

SINR [dB]	CQI code	Modulation	Code Rate	Spectral efficiency
-6.7	1	QPSK	0.076	0.15
-4.7	2	QPSK	0.12	0.23
-2.3	3	QPSK	0.19	0.38
0.2	4	QPSK	0.3	0.60
2.4	5	QPSK	0.44	0.88
4.3	6	QPSK	0.59	1.18
5.9	7	16QAM	0.37	1.48
8.1	8	16QAM	0.48	1.91
10.3	9	16QAM	0.6	2.41
11.7	10	64QAM	0.45	2.73
14.1	11	64QAM	0.55	3.32
16.3	12	64QAM	0.65	3.90
18.7	13	64QAM	0.75	4.52
21.0	14	64QAM	0.85	5.12
22.7	15	64QAM	0.93	5.55

A. Ghosh, A. R. Ratasuk, *Essentials of LTE and LTE-A*, Cambridge University Press, 2011.

4/5G Adaptive Modulation and Coding



A. Ghosh, A. R. Ratasuk, *Essentials of LTE and LTE-A*, Cambridge University Press, 2011.

RF bandwidth & spectral efficiency of M-QAM

- bit rate

$$R_b = R_s \log_2 M \tag{35}$$

- bandwidth with RC pulse

$$\Delta f_{RF} = (1+\alpha)R_s = (1+\alpha)\frac{R_b}{\log_2 M} \tag{36}$$

- spectral efficiency

$$\eta = \frac{R_b}{\Delta f_{RF}} = \frac{\log_2 M}{1+\alpha} \qquad \text{[bit/s/Hz]} \tag{37}$$

Demodulation of M-ary QAM

- M-QAM

$$M\text{-QAM} = (\sqrt{M}\text{-PAM}) \times (\sqrt{M}\text{-PAM})$$

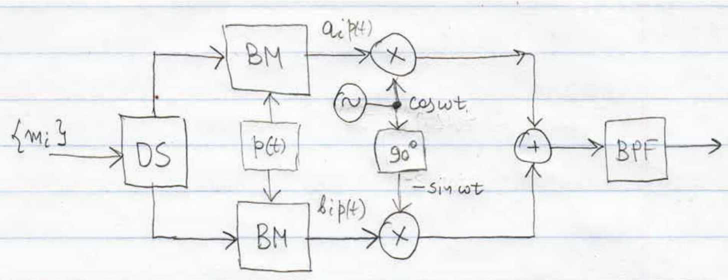
$$x(t) = m_I(t) A_c \cos\omega_c t - m_Q(t) A_c \sin\omega_c t \tag{38}$$

- demodulation:

$$\begin{aligned} m_I(t) &= LPF\{x(t)\cos\omega_c t\} \\ m_Q(t) &= -LPF\{x(t)\sin\omega_c t\} \end{aligned} \tag{39}$$

- add baseband demodulation of $m_I(t),\ m_Q(t)$ separately as $\sqrt{M}$-PAM

QAM Modulator

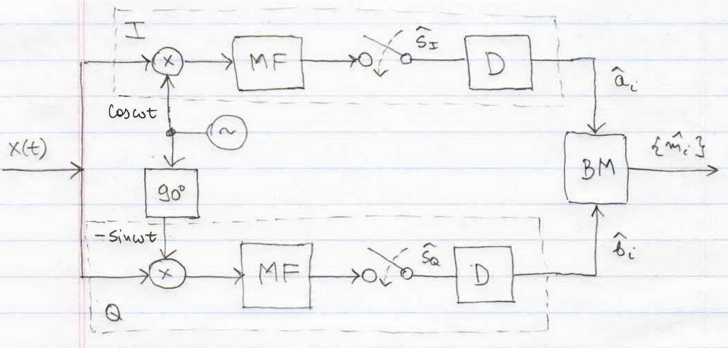


- DS = data splitter
- BPF = bandpass filter
- BM = $\sqrt{M}$-PAM baseband modulator

QAM Demodulator

- down-conversion, MF + decision
- QAM demodulator = I-PAM + Q-PAM demod.
- in practice: $M = 8, 16, 64, \ldots, 1024$.

QAM Demodulator



- MF = matched filter (for $p(t)$)
- BM = bit mapping, $(\hat{a}_i, \hat{b}_i) \to (0101\ldots)$
- D = detector/decision, $(\hat{s}_I, \hat{s}_Q) \to (\hat{a}_i, \hat{b}_i)$
- I, Q = in-phase and quadrature channels

Summary

- carrier modulation
  - BPSK, QPSK, QAM
- signal constellation
- modulators/demodulators
- bandwidth and spectral efficiency.
- BER, SER.

Reading

- S. Haykin, Digital Communication Systems, Wiley, 2014.
- J.M. Wozencraft, I.M. Jacobs, Principles of Communication Engineering, Wiley, 1965.
- R.E. Ziemer, W.H. Tranter, Principles of Communications, Wiley, New York, 2009.
- Other books (see the reference list).

Note: Do <u>not</u> forget to do end-of-chapter problems. Remember the learning efficiency pyramid!