ELG5375: Digital Comunication

Lecture 7

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Transmission over continuous-time channel

- system model:

$$m \;\to\; \boxed{\text{Tx}:\;\; s(t)} \;\to\; \boxed{\text{channel}} \;\to\; \boxed{\text{Rx}:\;\; x(t)} \;\to\; \widehat{m} \quad (1)$$

- continuous-time channel:

$$x(t) = s(t) + \xi(t),\; 0 \le t \le T \quad (2)$$

  - $s(t)$ = message-carrying signal (e.g. M-PAM)
  - $\xi(t)$ = additive noise (e.g. AWGN)
  - T = transmission interval

- performance metric: reliability, via P_e

$$P_e \triangleq \Pr\{\widehat{m} \neq m\},\; \widehat{m} = \mathrm{Rx}\{x(t)\} \quad (3)$$

- **optimal Rx**: minimize P_e

$$\min\; P_e \text{ via } \mathrm{Rx}\{\cdot\} \;\to\; \textbf{how}? \quad (4)$$

Transmission over continuous-time channel

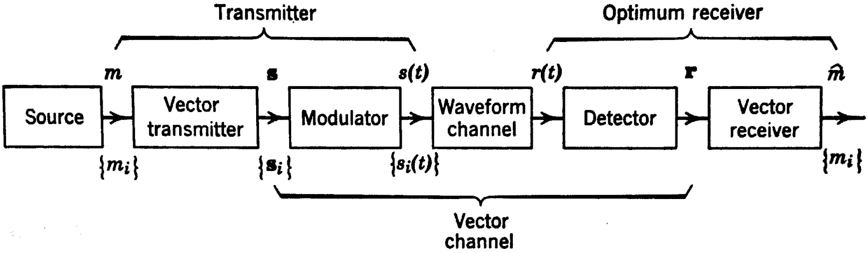


Figure 4.17 Reduction of waveform channel to vector channel. The modulator converts $\mathbf{s}$ to $s(t)$ by the mechanism of Fig. 4.12. The detector extracts the relevant received vector $\mathbf{r}$ from $r(t)$ by the mechanism of Fig. 4.16.

J.M. Wozencraft, I.M. Jacobs, Principles of Communication Engineering, Wiley, 1965.

Optimal Rx: how?

- optimal Rx: minimize P_e, but **how?**

- via signal space
- signals as vectors: from impossible problem to easy!
- geometric insights
- MAP/ML principle etc.

Signal space: geometric representation of signals

- continuous-time signals $\rightarrow$ vectors, via basis functions
- orthonormal basis functions $\{\phi_i(t)\}$

$$\int_T \phi_i(t)\phi_n^*(t)dt = \delta_{in} \triangleq \begin{cases} 1, & i = n; \\ 0, & i \neq n. \end{cases} \quad (5)$$

- $T = \infty$ is allowed
- any (reasonable) $s(t)$ can be expanded as

$$s(t) = \sum_n s_n \phi_n(t), \; s_n = \int_T s(t)\phi_n^*(t)dt \quad (6)$$

- i.e. $s(t) \;\Leftrightarrow\; \mathbf{s} = [s_1, s_2, ...]'$
- Examples: Fourier series, sampling series

Signal space: Fourier series

- periodic signal $s(t)$, of period T:

$$s(t) = \sum_{n=-\infty}^{\infty} s_n e^{jn\omega_0 t},\ s_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t} dt, \tag{7}$$

$\omega_0 = 2\pi f_0,\ f_0 = 1/T =$ fundamental frequency

- basis functions:

$$\phi_n(t) = e^{jn\omega_0 t} \tag{8}$$

- signal-space dimensionality: $K = \infty$
- **Q**: verify orthonormality

Signal space: sampling series/PAM

- continuous bandlimited signal $s(t)$, of bandwidth Δf:

$$s(t) = \sum_{n=-\infty}^{\infty} s(nT_s)\text{sinc}\left(\frac{t}{T_s} - n\right) \tag{9}$$

  - T_s is the sampling interval, $T_s = 1/f_s$
  - f_s is the sampling frequency, $f_s \geq 2\Delta f$

- basis functions and s_n:

$$\phi_n(t) = \frac{1}{\sqrt{T_s}}\text{sinc}\left(\frac{t}{T_s} - n\right),\ s_n = \sqrt{T_s}s(nT_s) \tag{10}$$

- signal-space dimensionality: $K = \infty$
- **Q**: verify orthonormality and s_n

Signal space: M-PAM

- single-pulse M-PAM

$$s(t) = a \cdot p(t) \;\rightarrow\; \phi_1(t) = \frac{1}{\sqrt{E_p}} p(t),\; s_1 = a\sqrt{E_p} \qquad (11)$$

- e.g. $s_1 = \pm\sqrt{E_p}$ if $a = \pm 1$ (2-PAM)
- signal-space dimensionality: $K = 1$ (for any M)

Signal space: M-PAM

- N orthogonal pulses of M-PAM

$$s(t) = \sum_{i=1}^{N} a_i p(t - iT), \tag{12}$$

$$\int_{-\infty}^{\infty} p(t - iT)p(t - nT)dt = 0 \ \forall \ i \neq n \tag{13}$$

- basis functions: $n = 1...N$

$$\phi_n(t) = \frac{1}{\sqrt{E_p}} p(t - nT), \ s_n = \int_{-\infty}^{\infty} s(t)p(t - nT)dt \tag{14}$$

- signal-space dimensionality: $K = N$ (for any M)
- example: non-overlapping rectangular pulses

Signal space: M-QAM

- single-pulse M-QAM of duration $T = kT_c$

$$
\begin{aligned}
s(t) &= a \cdot \cos\omega_c t + b \cdot \sin\omega_c t \qquad (15) \\
&\rightarrow\ \phi_1(t) = E^{-1/2}\cos\omega_c t,\ s_1 = a\sqrt{E},\ E = T/2 \\
&\rightarrow\ \phi_2(t) = E^{-1/2}\sin\omega_c t,\ s_2 = b\sqrt{E}
\end{aligned}
$$

- signal-space dimensionality: $K = 2$
- **Q:** verify orthonormality and s_n

Signals as vectors → signal space

- continuous-time signals → vectors, via basis functions
- energy = distance (Euclidean norm) squared

$$E_s \triangleq \int_T |s(t)|^2 dt = \sum_n |s_n|^2 \triangleq |\mathbf{s}|^2 \tag{16}$$

- scalar product

$$\int_T s_1(t) s_2^*(t) dt = \sum_n s_{1n} s_{2n}^* \triangleq \mathbf{s}_2^+ \mathbf{s}_1 \tag{17}$$

- addition/subtraction etc.:

$$s_1(t) \pm s_2(t) \;\Leftrightarrow\; \mathbf{s}_1 \pm \mathbf{s}_2 \tag{18}$$

- signal-space dimensionality: N if $n = 1...N$; $s_n = 0 \; \forall n > N$

Signals as vectors $\rightarrow$ signal space

- angle, from $\mathbf{s}_2^+\mathbf{s}_1 = \cos\theta|\mathbf{s}_2||\mathbf{s}_1|$

$$\cos\theta = \frac{\mathbf{s}_2^+\mathbf{s}_1}{|\mathbf{s}_2||\mathbf{s}_1|} = \frac{1}{\sqrt{E_{s2}E_{s1}}}\int_T s_1(t)s_2^*(t)dt \tag{19}$$

- orthogonality of signals:

$$\cos\theta = 0 \;\Leftrightarrow\; \mathbf{s}_2^+\mathbf{s}_1 = 0 \;\Leftrightarrow\; \int_T s_1(t)s_2^*(t)dt = 0 \tag{20}$$

Signals as vectors: Cauchy-Schwartz inequality

- fundamental tool, for optimal Rx and many more ...
- **Cauchy-Schwartz inequality**:

$$\left| \int_T s_1(t) s_2^*(t) dt \right|^2 \leq \int_T |s_1(t)|^2 dt \int_T |s_2(t)|^2 dt \qquad (21)$$

- equality iff $s_2(t) = a \cdot s_1^*(t)$
- proof: from $|\cos\theta| \leq 1$

Continuous-time channel → vector channel

- via signal space, for signal and noise
- replace time-domain signals by vectors:

$$r(t) = s(t) + \xi(t) \;\rightarrow\; \mathbf{r} = \mathbf{s} + \boldsymbol{\xi} \tag{22}$$

where

$$s(t) = \sum_{n=1}^{K} s_n \phi_n(t) \;\rightarrow\; \mathbf{s} = [s_1, s_2, ..]' \tag{23}$$

$$\xi(t) = \sum_{n=1}^{\infty} \xi_n \phi_n(t) \rightarrow\; \boldsymbol{\xi} = [\xi_1, \xi_2, ...]' \tag{24}$$

- $s(t)$: $n = 1...K$, $K =$ signal-space dimensionality
- $\xi(t)$: $n = 1...\infty$, noise dim. $= \infty$, can be tricky

Transmission over continuous-time channel in signal space

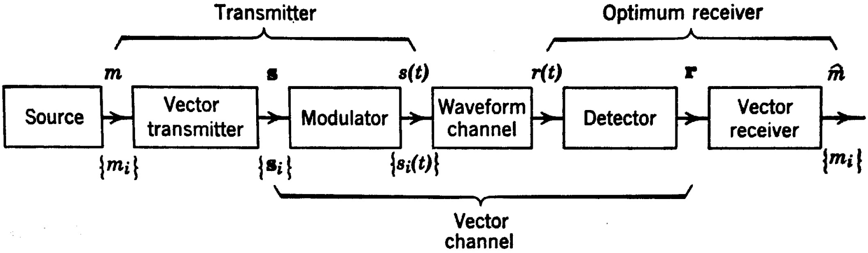


Figure 4.17 Reduction of waveform channel to vector channel. The modulator converts $\mathbf{s}$ to $s(t)$ by the mechanism of Fig. 4.12. The detector extracts the relevant received vector $\mathbf{r}$ from $r(t)$ by the mechanism of Fig. 4.16.

J.M. Wozencraft, I.M. Jacobs, Principles of Communication Engineering, Wiley, 1965.

Optimal Rx for vector channel

- use vector channel

$$\mathbf{r} = \mathbf{s} + \boldsymbol{\xi} \tag{25}$$

- where $\mathbf{s}$ carries the message: $m \leftrightarrow \mathbf{s}$ or $\mathbf{s} = \mathbf{s}(m)$
- and apply MAP decision rule $\rightarrow$ **optimal Rx**:

$$\widehat{\mathbf{s}} = \arg\max_{\mathbf{s}} p(\mathbf{r}|\mathbf{s})p(\mathbf{s}) \ \leftrightarrow \ \widehat{m} = \arg\max_{m} p(\mathbf{r}|m)p(m) \tag{26}$$

- (almost) trivial!

Signal Space Projection

- project $r(t)$ on signal space
- 2 approaches in the literature, same final result
- both use orthonormal basis functions $\{\phi_i(t)\}$

$$\int_T \phi_i(t)(t)\phi_n^*(t)dt = \delta_{in} \triangleq \begin{cases} 1, & i = n; \\ 0, & i \neq n. \end{cases} \tag{27}$$

  and replace time-domain signals with vectors
- $T = \infty$ is allowed
- noise expansion can be tricky

Approach 1

- project $s(t)$ and $\xi(t)$ onto $\{\phi_i(t)\}_{i=1}^{\infty}$:

$$s_i = \int_T s(t)\phi_i^*(t)dt \;\Leftrightarrow\; s(t) = \sum_{i=1}^{K} s_i\phi_i(t), \tag{28}$$

$$\xi_i = \int_T \xi(t)\phi_i^*(t)dt \;\Leftrightarrow\; \xi(t) = \sum_{i=1}^{\infty} \xi_i\phi_i(t), \tag{29}$$

where $K =$ signal space dimensionality, noise dim. $= \infty$.

$$r(t) = \sum_{i=1}^{K}(s_i + \xi_i)\phi_i(t) + \sum_{i=K+1}^{\infty} \xi_i\phi_i(t) \tag{30}$$

- issue: convergence of the noise series

Approach 2

- finite number N of observations (measurements) in the physical world
- "naked" noise is not observable, only via LTI measurements
- therefore, the Rx noise is

$$\xi(t) = \sum_{i=1}^{N} \xi_i \phi_i(t), \tag{31}$$

where observable noise dim. $= N \gg K$ (can be very large but still finite), and

$$r(t) = \sum_{i=1}^{K} (s_i + \xi_i)\phi_i(t) + \sum_{i=K+1}^{N} \xi_i \phi_i(t) \tag{32}$$

- no convergence issues
- Approach 2 $\rightarrow$ Approach 1 via $N \rightarrow \infty$

Properties for AWGN

- in both cases, ξ_i and ξ_n are independent,

$$\xi_i \sim N(0, \sigma_0^2), \text{ iid } \rightarrow p(\xi_i, \xi_n) = p(\xi_i)p(\xi_n), \; i \neq n \tag{33}$$

- the pdf of $\boldsymbol{\xi} = [\xi_1, \xi_2 ... \xi_N]'$:

$$p(\boldsymbol{\xi}) = (2\pi\sigma_0^2)^{-N/2} e^{-|\boldsymbol{\xi}|^2/(2\sigma_0^2)} \tag{34}$$

- key for optimal Rx decision and analysis
- turns problem from (almost) impossible to (almost) trivial
- works for general M-ary modulation, not only binary

Q: prove that $\widetilde{\xi}(t) = \xi(t) - \xi_K(t)$ is independent of $\xi_K(t_0)$ for any t, t_0, K, N, where

$$\xi_K(t) = \sum_{i=1}^{K} \xi_i \phi_i(t), \; \widetilde{\xi}(t) = \sum_{i=K+1}^{N} \xi_i \phi_i(t) \tag{35}$$

Signal space representation for AWGN channel

- $\xi(t) =$ AWGN,

$$r(t) = s(t) + \xi(t) = \sum_{i=1}^{K} (s_i + \xi_i)\phi_i(t) + \sum_{i=K+1}^{N} \xi_i \phi_i(t) \qquad (36)$$

- replace time-domain signals by vectors:

$$s(t) = \sum_{i=1}^{K} s_i \phi_i(t) \; \rightarrow \; \mathbf{s} = [s_1, .., s_K]' \qquad (37)$$

$$\xi(t) = \sum_{i=1}^{K} \xi_i \phi_i(t) + \sum_{i=K+1}^{N} \xi_i \phi_i(t) \; \rightarrow \; \boldsymbol{\xi} = [\xi_1, \xi_2, ...\xi_N]'$$

From time domain to signal space

- our approach: project $r(t)$ on $\{\phi_i(t)\}_{i=1}^{K}$, discard the rest:

$$
\begin{aligned}
r(t) &= s(t) + \xi(t) = \sum_{i=1}^{K} (s_i + \xi_i)\phi_i(t) + \sum_{i=K+1}^{N} \xi_i \phi_i(t) \\
&= r_1(t) + r_2(t) \\
&\to \ \mathbf{r}_1 = [s_1 + \xi_1, ..., s_K + \xi_K]', \mathbf{r}_2 = [\xi_{K+1}, \xi_{K+2}, ...\xi_N]' \qquad (38) \\
&\to \ \mathbf{r} = \mathbf{s} + \boldsymbol{\xi} = [\mathbf{r}_1', \mathbf{r}_2']'
\end{aligned}
$$

- discard $\mathbf{r}_2, r_2(t)$ (irrelevant data)
- **Q: why?**

Sufficient statistics in AWGN channel

- Sufficient statistics:

$$p(\mathbf{r}_1, \mathbf{r}_2|\mathbf{s}) = p(\mathbf{r}_1|\mathbf{s})p(\mathbf{r}_2|\mathbf{r}_1, \mathbf{s}) = p(\mathbf{r}_1|\mathbf{s})p(\mathbf{r}_2) \tag{39}$$

  so that

$$\arg\max_{\mathbf{s}} p(\mathbf{r}_1, \mathbf{r}_2|\mathbf{s})p(\mathbf{s}) = \arg\max_{\mathbf{s}} p(\mathbf{r}_1|\mathbf{s})p(\mathbf{s}) \tag{40}$$

  i.e. $\mathbf{r}_2, r_2(t)$ is irrelevant data.
- Optimal (MAP) Rx: based on $\mathbf{r}_1$ only, no use for $\mathbf{r}_2, r_2(t)$.
- Significant reduction in dimensionality: $K \ll N$

Optimal (MAP) Rx

- MAP decision rule $\rightarrow$ min P_e:

$$\widehat{\mathbf{s}} = \arg\max_{\mathbf{s}} p(\mathbf{r}_1|\mathbf{s})p(\mathbf{s}) \ \rightarrow \ \widehat{m} \tag{41}$$

- implementation?

Optimal (ML) Rx in AWGN

- **ML decision rule**:

$$\hat{\mathbf{s}} = \arg\max_{\mathbf{s}_m} p(\mathbf{r}_1|\mathbf{s}_m) \;= \arg\min_{\mathbf{s}_m} |\mathbf{r}_1 - \mathbf{s}_m|^2 \tag{42}$$

- i.e. **min-distance rule**: select $\mathbf{s}_m$ closest to $\mathbf{r}_1$

$$\boxed{\text{ML decision rule } = \text{ min-distance rule}} \tag{43}$$

- equivalently

$$\hat{\mathbf{s}} = \arg\max_{\mathbf{s}_m} \left(2\text{Re}\{\mathbf{s}_m^+\mathbf{r}_1\} - |\mathbf{s}_m|^2\right) \tag{44}$$

- if equal energy: $E_s = |\mathbf{s}_m|^2$ far all m,

$$\hat{\mathbf{s}} = \arg\max_{\mathbf{s}_m} \text{Re}\{\mathbf{s}_m^+\mathbf{r}_1\} \tag{45}$$

- **Q:** prove all of the above

Optimal (ML) Rx in AWGN

- implementation of **ML decision rule**

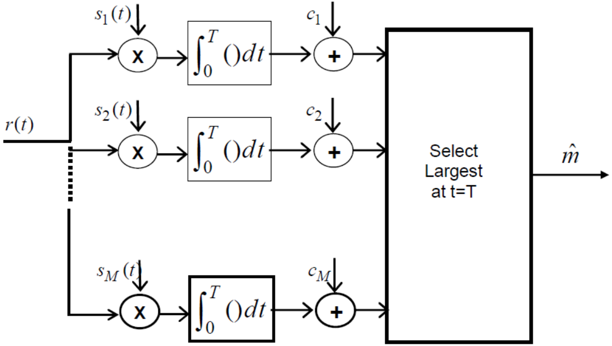
$$\widehat{\mathbf{s}} = \arg\max_{\mathbf{s}_m} \mathrm{Re}\{\mathbf{s}_m^+ \mathbf{r}_1\} = \arg\max_{\mathbf{s}_m} \int_T r(t) s_m^*(t) dt \qquad (46)$$

- i.e. **correlator Rx**:

$$r_m = \int_T r(t) s_m^*(t) dt \ \rightarrow \ \widehat{m} = \arg\max_m r_m \qquad (47)$$

- **Q:** prove 2nd equality in (46)

Correlator Rx



c_m adjusts for unequal symbol energy: $c_m = -E_m/2 = -|\mathbf{s}_m|^2/2$

Matched Filter (MF) Rx

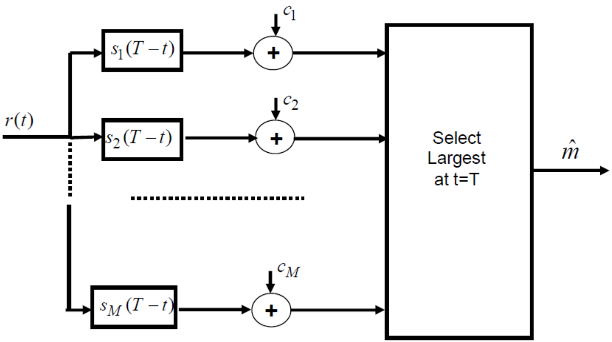
- implementation via **matched filter** $h_m(t) = s_m^*(T-t)$

$$r_m = \int_T r(t)s_m^*(t)dt = r(t) * h_m(t)|_{t=T} \tag{48}$$

- **Q:** prove this
- i.e., MF = signal-space projector

- **optimality** of the Rx BD of Lec. 5 is now **fully proved**
- no any other Rx has smaller P_e

Matched Filter (MF) Rx



c_m adjusts for unequal symbol energy: $c_m = -E_m/2 = -|\mathbf{s}_m|^2/2$

Signal-space projector/correlator Rx

ML decision rule: $\widehat{m} = \arg\max_m \left(2\text{Re}\{\mathbf{s}_m^+ \mathbf{r}_1\} - |\mathbf{s}_m|^2\right)$

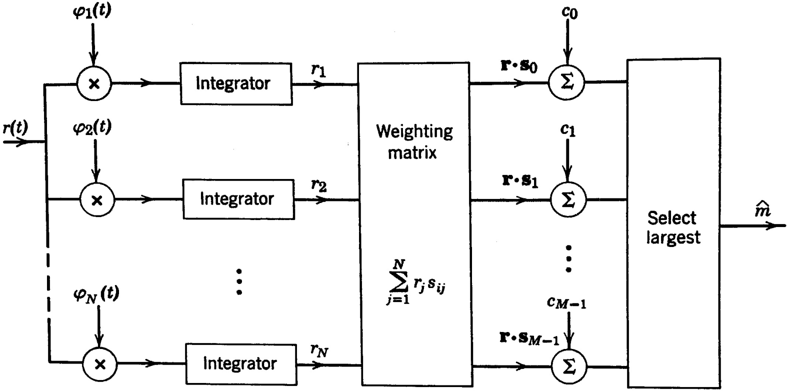


Figure 4.18 Diagram of the correlation receiver. The bias terms $\{c_i\}$ are given by Eq. 4.53b.

J.M. Wozencraft, I.M. Jacobs, Principles of Communication Engineering, Wiley, 1965.

Signal-space MF Rx

ML decision rule: $\widehat{m} = \arg\max_m \left(2\text{Re}\{\mathbf{s}_m^+ \mathbf{r}_1\} - |\mathbf{s}_m|^2\right)$

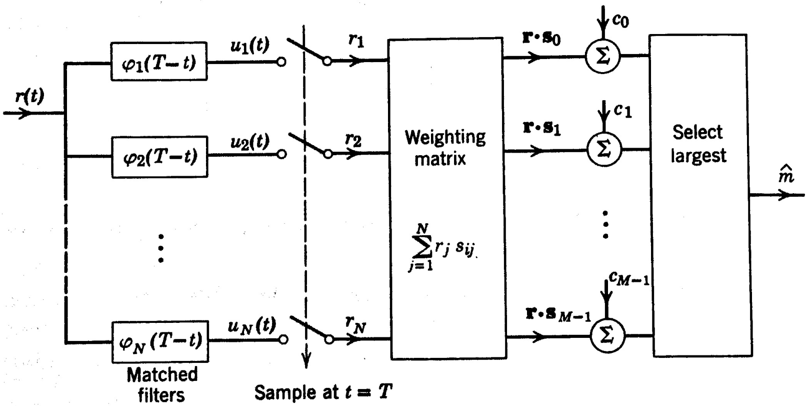


Figure 4.19 Diagram of the matched-filter receiver.

J.M. Wozencraft, I.M. Jacobs, Principles of Communication Engineering, Wiley, 1965.

Example: 2-PAM, rect. pulse, $m = \pm 1$

- system/channel model:

$$r(t) = mp(t) + \xi(t),\ p(t) = A\,\Pi(t/T) \tag{49}$$

- matched filter (MF):

$$h(t) = \alpha p(T - t) = T^{-1}\Pi(t/T) \tag{50}$$

- sufficient statistics = MF output at $t = T$:

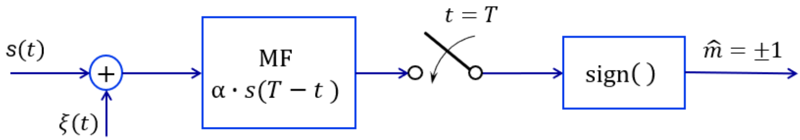
$$r = \frac{1}{T}\int_0^T r(t)dt \tag{51}$$

- optimal decision:

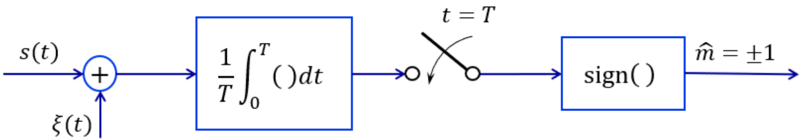
$$\widehat{m}(r) = \mathrm{sign}(r) = \begin{cases} 1, & \text{if } r > 0 \\ -1, & \text{otherwise} \end{cases} \tag{52}$$

Example: optimum Rx BD

- MF Rx:



- Integrator Rx:



- nothing better exists

Example: performance analysis

- MF output sample: $r = r_s + r_n$
- signal r_s and noise r_n samples:

$$r_s = \frac{1}{T}\int_0^T mp(t)dt = mA,\;\; r_n = \frac{1}{T}\int_0^T \xi(t)dt \tag{53}$$

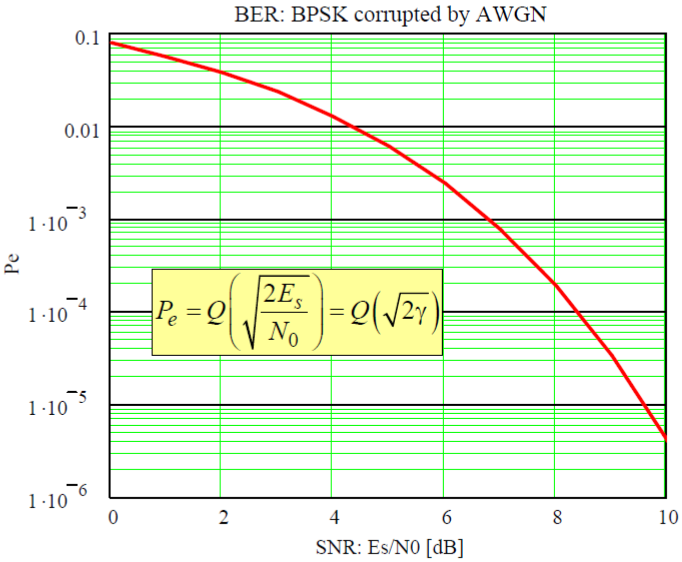
- noise sample properties: $r_n \sim \mathcal{N}(0, \sigma_n^2)$

$$\bar{r}_n = 0,\;\; \mathrm{var}\{r_n\} = \sigma_n^2 = \sigma_0^2/T,\;\; \sigma_0^2 = N_0/2 \tag{54}$$

- SNR & BER:

$$\gamma = \frac{E_s}{N_0} = \frac{A^2 T}{N_0},\quad P_e = Q\left(\sqrt{\frac{A^2 T}{\sigma_0^2}}\right) = Q\left(\sqrt{2\gamma}\right) \tag{55}$$

Example: performance analysis, BER



Summary

- signal space
- continuous-time signals $\rightarrow$ vectors
- optimal Rx: from impossible to (almost) trivial!
- MAP/ML/min. distance rules
- optimal Rx = MF Rx + sampling
- optimal Rx BD: fully proved

Reading

- S. Haykin, Digital Communication Systems, Wiley, 2014. Ch. 7.1–7.6.
- J.M. Wozencraft, I.M. Jacobs, Principles of Communication Engineering, Wiley, 1965. Ch.4.
- H.L. Van Trees, Detection, Estimation, and Modulation Theory, Part I, Wiley, 2001 (original 1968). Ch. 4.1–4.2.
- A. Lapidoth, A Foundation in Digital Communication, Cambridge University Press, 2017. Ch. 26.