ELG5375: Digital Communications

Lecture 4

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How to send 1 bit?

- consider most simple case: distortionless
  - no noise/interference
  - no any other distortions (e.g., ISI)
- key idea: encode bit b into signal amplitude a
- **pulse-amplitude modulation (PAM)**

$$b \;\rightarrow\; \boxed{\text{Tx}: \; x(t) = a \cdot p(t)} \;\rightarrow\; \boxed{\text{Rx}: \; y(t) = x(t)} \;\rightarrow\; \hat{b} \qquad (1)$$

- bit mapping: $b \;\rightarrow\; a$ [V]

$$\begin{aligned} b = 1 \;&\rightarrow\; a = A_1 \\ b = 0 \;&\rightarrow\; a = A_0 \end{aligned} \qquad (2)$$

- $p(t)$ = pulse shape/waveform

How to send 1 bit: binary PAM (2-PAM)

$$b \;\rightarrow\; x(t) = a \cdot p(t) \;\rightarrow\; y(t) = x(t) \;\rightarrow\; \hat{b} \tag{3}$$

- normalize $p(t)$: $p(0) = 1$
- how to recover b at Rx?

How to send 1 bit: binary PAM (2-PAM)

$$b \;\rightarrow\; x(t) = a \cdot p(t) \;\rightarrow\; y(t) = x(t) \;\rightarrow\; \hat{b} \tag{3}$$

- normalize $p(t)$: $p(0) = 1$
- how to recover b at Rx?

$$y(0) = a \cdot p(0) = a \;\rightarrow\; \hat{b} = \begin{cases} 1, & \text{if } a = A_1 \\ 0, & \text{if } a = A_0 \end{cases} \tag{4}$$

- key: measure $y(t)$ at $t = 0$ (or any other suitable t)
- distortionless: $\hat{b} = b$ (no errors)
- pulse $p(t)$: anything (reasonable)

How to send N bits?

- sequential transmission

$$\{b_i\} \;\rightarrow\; x(t) = \sum_{i=0}^{N-1} a_i p(t - iT_s) \tag{5}$$

- i = discrete time
- $T_s = T_b$ = pulse/bit interval (duration) [s]
- $R_b = 1/T_b$ = bit rate [bit/s]
- $R_s = 1/T_s$ = symbol rate [sym/s]

$$R_s = R_b \text{ for 2-PAM} \tag{6}$$

- bit mapping: $b_i \;\rightarrow\; a_i$
- how to recover $\{b_i\}$ at Rx?

How to send N bits?

- how to recover b_k at Rx?

$$y(kT_s) = a_k + \underbrace{\sum_{i \neq k} a_i p((k-i)T_s)}_{\text{ISI}} \tag{7}$$

- ISI = inter-symbol interference
- **zero-ISI condition**:

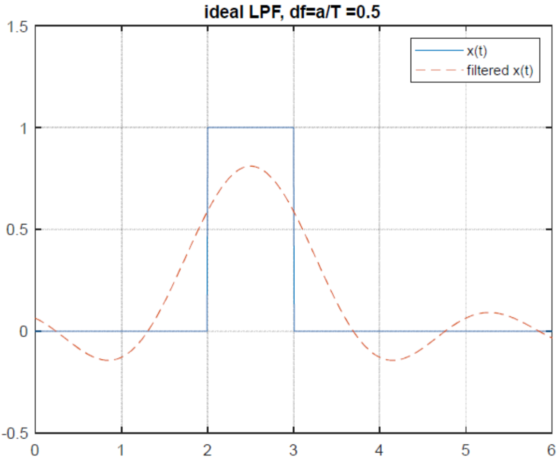
$$p(iT_s) = 0 \ \forall \ i = \pm 1, \pm 2, ... \tag{8}$$

- so that

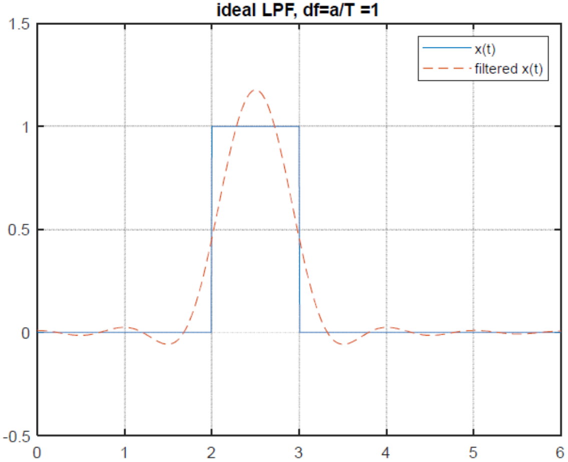
$$y(kT_s) = a_k \ \rightarrow \ b_k \tag{9}$$

- **zero-ISI: how?**

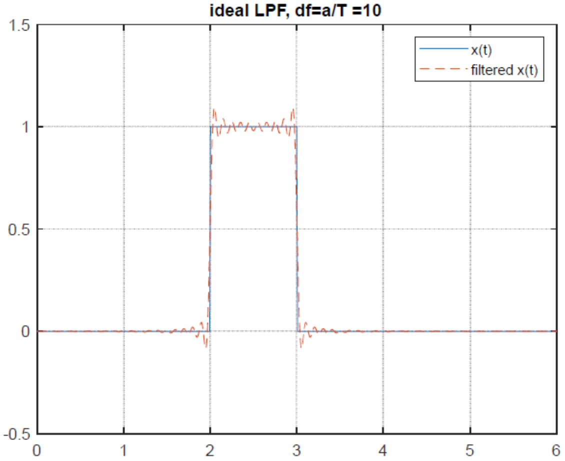
ISI: an example with ideal LPF



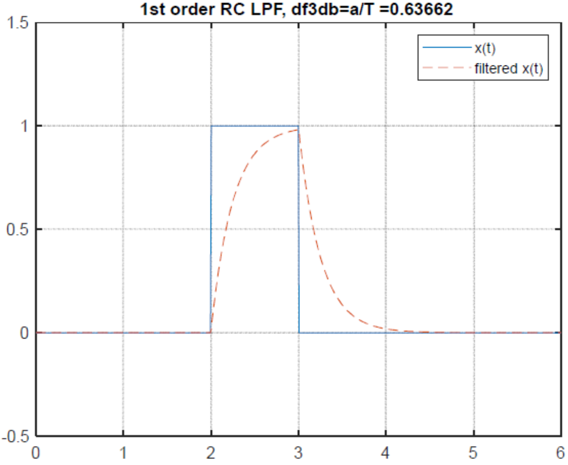
ISI: an example with ideal LPF



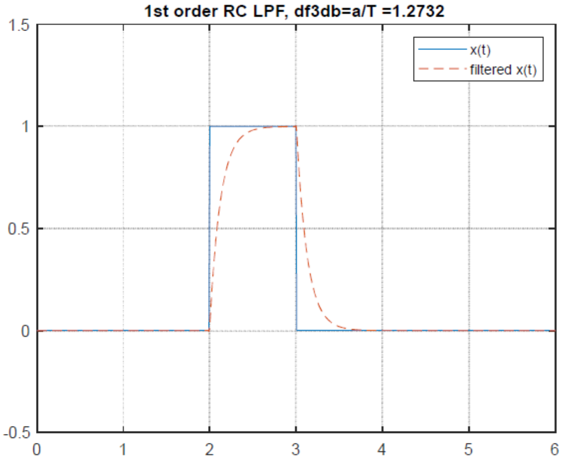
ISI: an example with ideal LPF



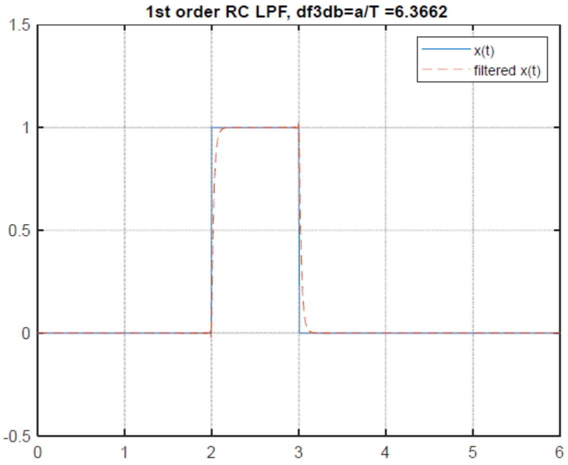
ISI: an example with real RC LPF



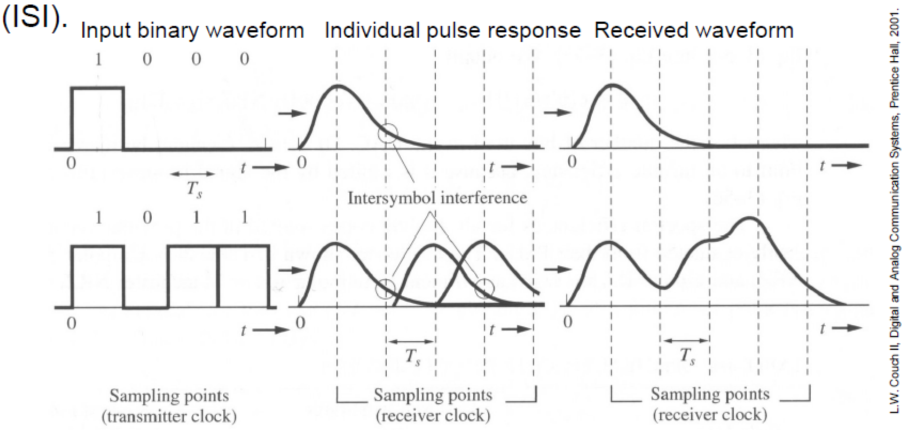
ISI: an example with real RC LPF



ISI: an example with real RC LPF



Sequential transmission with ISI



Zero-ISI: how?

- zero-ISI: Nyquist's 1st criterion[1]

$$p(iT_s) = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0 \end{cases} \qquad (10)$$

- how to find $p(t)$?
- Nyquist'24:

$$\frac{1}{T_s} \sum_{n=-\infty}^{\infty} S_p(f - nR_s) = 1 \ \forall f \qquad (11)$$

- $S_p(f) = \text{FT}\{p(t)\}$
- $R_s = 1/T_s =$ symbol rate [sym/s]

[1] H. Nyquist, Certain Factors Affecting Telegraph Speed, Bell System Technical Journal, Apr. 1924.

Zero-ISI for bandlimited channel/system

- Δf = channel/system bandwidth, $\Delta f < \infty$
- **Q:** would rectangular pulse work? why?

Zero-ISI for bandlimited channel/system

- Δf = channel/system bandwidth, $\Delta f < \infty$
- **Q:** would rectangular pulse work? why?

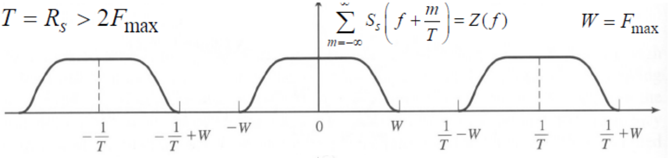
- Nyquist **zero-ISI criterion**:

  1. $R_s > 2\Delta f \;\to$ impossible
  2. $R_s = 2\Delta f \;\to\; p(t) = \text{sinc}(t/T_s)$, unique
  3. $R_s < 2\Delta f \;\to$ many, e.g. raised-cosine (RC)

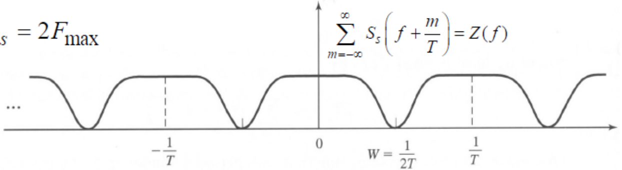
- $2\Delta f$ = **Nyquist rate**
- zero-ISI *does not imply* that pulses do not overlap!
- i.e. zero ISI at sampling times only

Zero-ISI for bandlimited channel/system

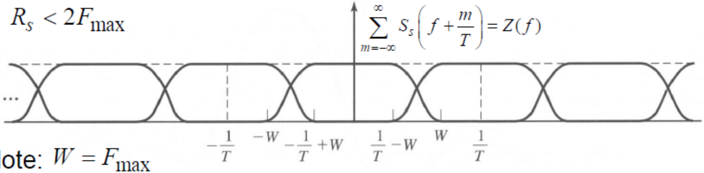
Case 1: $1/T = R_s > 2F_{\max}$



Case 2: $R_s = 2F_{\max}$

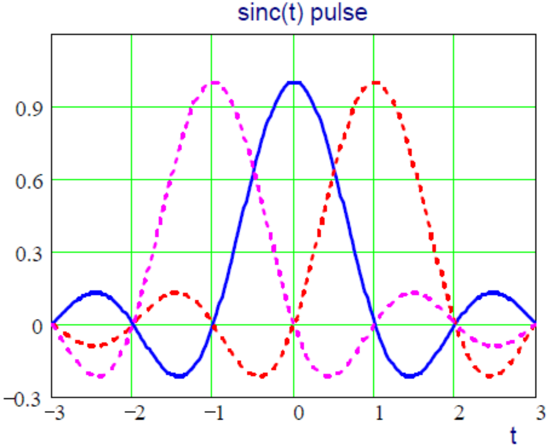


Case 3: $R_s < 2F_{\max}$



Note: $W = F_{\max}$

Zero-ISI for bandlimited channel/system



Harry Nyquist

Certain Factors Affecting Telegraph Speed[1]

By H. NYQUIST

SYNOPSIS: This paper considers two fundamental factors entering into the maximum speed of transmission of intelligence by telegraph. These factors are signal shaping and choice of codes. The first is concerned with the best wave shape to be impressed on the transmitting medium so as to permit of greater speed without undue interference either in the circuit under consideration or in those adjacent, while the latter deals with the choice of codes which will permit of transmitting a maximum amount of intelligence with a given number of signal elements.

It is shown that the wave shape depends somewhat on the type of circuit over which intelligence is to be transmitted and that for most cases the optimum wave is neither rectangular nor a half cycle sine wave as is frequently used but a wave of special form produced by sending a simple rectangular wave through a suitable network. The impedances usually associated with telegraph circuits are such as to produce a fair degree of signal shaping when a rectangular voltage wave is impressed.

Consideration of the choice of codes show that while it is desirable to use those involving more than two current values, there are limitations which prevent a large number of current values being used. A table of comparisons shows the relative speed efficiencies of various codes proposed. It is shown that no advantages result from the use of a sine wave for telegraph transmission as proposed by Squier and others[2] and that their arguments are based on erroneous assumptions.

SIGNAL SHAPING

SEVERAL different wave shapes will be assumed and comparison will be made between them as to:

1. Excellence of signals delivered at the distant end of the circuit, and
2. Interfering properties of the signals.

Consideration will first be given to the case where direct-current impulses are transmitted over a distortionless line, using a limited range of frequencies. Transmission over radio and carrier circuits will next be considered. It will be shown that these cases are closely related to the preceding one because of the fact that the transmitting medium in the case of either radio or carrier circuits closely approximates a distortionless line. Telegraphy over ordinary land lines

[1] Presented at the Midwinter Convention of the A. I. E. E., Philadelphia, Pa. February 4-8, 1924, and reprinted from the Journal of the A. I. E. E. Vol. 43, p. 124, 1924.

Born: 7 Feb. 1889, Värmland, Sweden
Died: 4 Apr. 1976 (aged 87), Texas, US

https://en.wikipedia.org/wiki/Harry_Nyquist

Maximizing Data Rate

- maximize $R_b = R_s$ s.t. zero ISI:

$$\boxed{\max R_s \text{ s.t. no ISI}} \rightarrow R_s = 2\Delta f,\ p(t) = \text{sinc}(tR_s) \qquad (12)$$

- constrained optimization via Nyquist criterion
- sinc pulse as the (unique) solution of the problem
- max rate = Nyquist rate $2\Delta f$
- **max. rate is bounded by system bandwidth**

Maximizing Data Rate

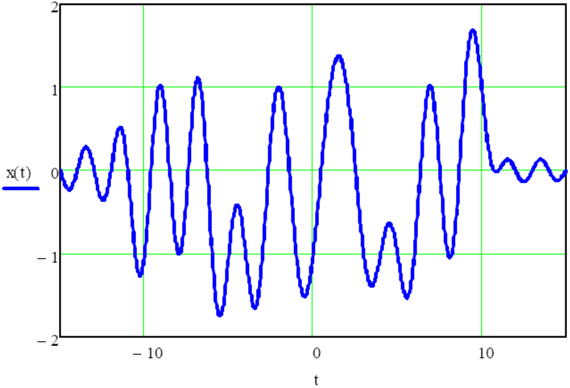
- Tx signal - sampling theorem reversed:

$$x(t) = \sum_{k=0}^{N-1} a_k \text{sinc}\left(\frac{t}{T_s} - k\right), \ T_s = \frac{1}{2\Delta f} \tag{13}$$

- a_k are not samples, but encoded bits!
- bandwidth: $\Delta f_x = \Delta f$
- also works for M-PAM
- $x(t)$ is a random signal/process, since a_k are random

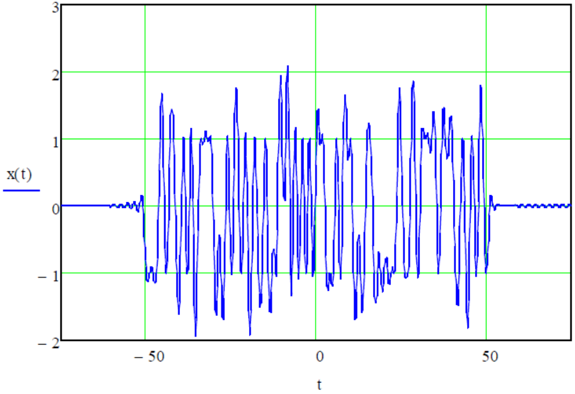
Sequential sinc-pulse transmission of random data

random data -> random signal $x(t) = \sum_{n=-10}^{10} a_n \text{sinc}(t-n), \;\; a_n = \pm 1$



Sequential sinc-pulse transmission of random data

random data -> random signal $x(t) = \sum_{n=-50}^{50} a_n \text{sinc}(t-n), \ \ a_n = \pm 1$



Max. rate transmission: an example

- $\Delta f = 20$ MHz, 2-PAM
- max rate s.t. zero-ISI?

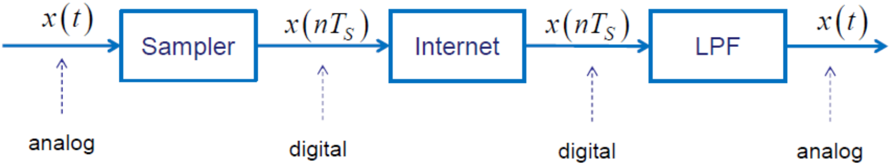
$$\max R_b = \max R_s = 2\Delta f = 40 \text{ [Mb/s]} \quad (14)$$

- and for wireless ? (BPSK)

$$\max R_b = \max R_s = \Delta f = 20 \text{ [Mb/s]} \quad (15)$$

Zero-ISI via Sampling Theorem

- Standard view: analog – digital - analog

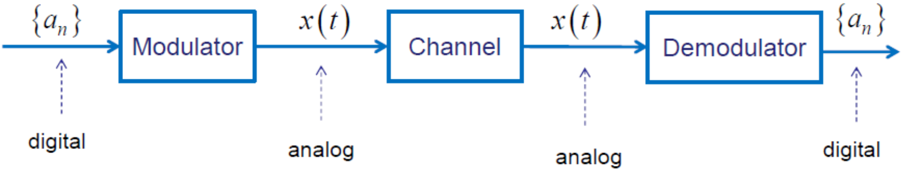


$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_S)\,\mathrm{sinc}\left(\frac{t}{T_S} - n\right)$$

$$\Delta f_x \le \frac{1}{2} f_s$$

Zero-ISI via Sampling Theorem

- PAM: digital - analog - digital

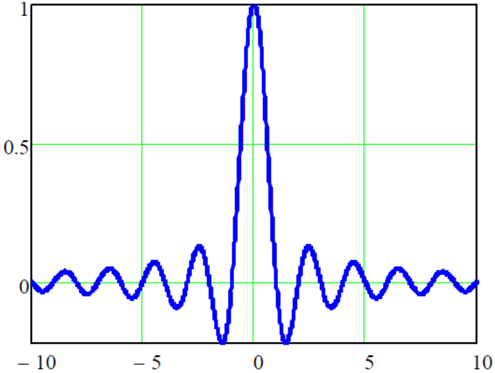


$$x(t) = \sum_{n=-\infty}^{\infty} a_n \text{sinc}\left(\frac{t}{T_S} - n\right)$$

$$\Delta f_x \leq \frac{1}{2} R_s$$

sinc pulse drawbacks

- sinc pulse has 2 practical (and serious) drawbacks



Raised cosine (RC) pulse

- sinc pulse has 2 practical drawbacks
- raised-cosine (RC) pulse partially overcomes those
- used extensively as a model of practical pulses

Raised cosine (RC) pulse

- RC spectrum (FT): 3 distinct regions

1. **flat**: $0 \leq |f| \leq f_1$

$$S_{rc}(f) = T_s \tag{16}$$

2. **transition**: $f_1 \leq |f| \leq f_2$

$$S_{rc}(f) = \left(1 + \cos\left[\pi T_s \alpha^{-1}(|f| - f_1)\right]\right) T_s/2 \tag{17}$$

3. **zero**: $|f| \geq f_2$

$$S_{rc}(f) = 0 \tag{18}$$

- f_1, f_2 are low/high roll-off (transition) frequencies:

$$f_1 \triangleq (1-\alpha)R_s/2, \;\; f_2 \triangleq (1+\alpha)R_s/2, \tag{19}$$

- α is roll-off factor, $0 \leq \alpha \leq 1$

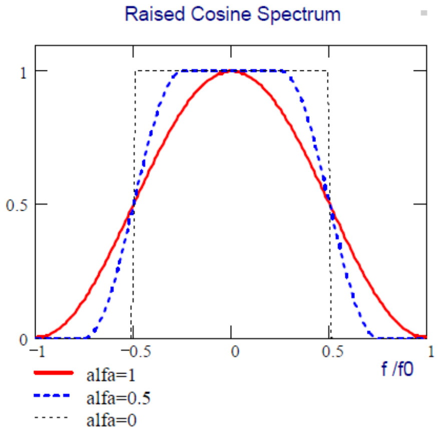
Raised cosine (RC) pulse

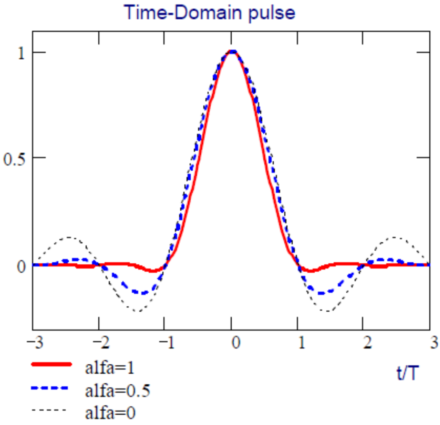
- transition region: "glues" flat and zero regions
- RC pulse bandwidth:

$$\Delta f_{rc} = f_2 = \frac{1+\alpha}{2} R_s \tag{20}$$

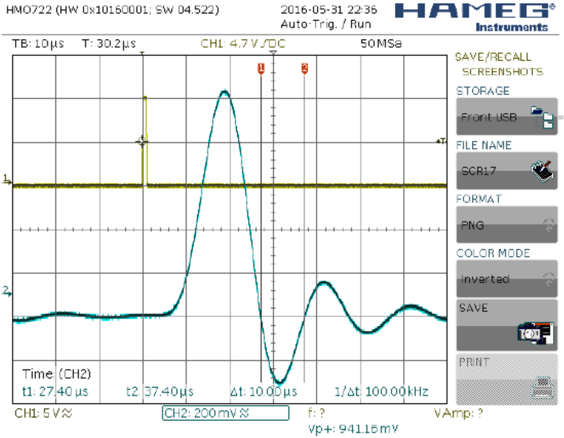
- RC pulse reduces to sinc if $\alpha = 0$ (so that $f_1 = f_2 = R_s$)
- excess bandwidth (above sinc): $\alpha R_s/2$, needed for smooth transition

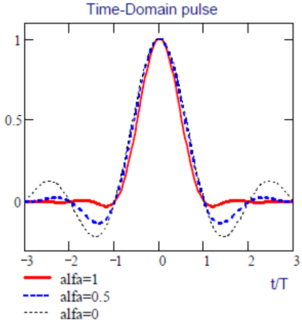
Raised cosine (RC) pulse/spectrum





Practical sinc (measured in lab)





Raised cosine (RC) pulse

- in time domain: via inverse FT

$$p_{rc}(t) = \text{sinc}\left(\frac{t}{T_s}\right) \frac{\cos(\pi\alpha t/T_s)}{1-(2\alpha t/T_s)^2} \tag{21}$$

- its shape is closer to practical pulses
- note that, for large t, its peaks scale as

$$|p_{rc}(t)| \sim \left(\frac{T_s}{t}\right)^3 \tag{22}$$

- so that small sampling time errors are not catastrophic

Maximizing rate with RC pulse

- maximum rate for zero ISI and given $\Delta f,\ \alpha$:

$$\boxed{\max R_s \text{ s.t. no ISI}, \Delta f, \alpha} \ \Rightarrow\ \max R_s = \frac{2\Delta f}{1+\alpha} \tag{23}$$

- via

$$\Delta f_{rc} = \frac{1+\alpha}{2} R_s \leq \Delta f \ \Rightarrow R_s \leq \frac{2\Delta f}{1+\alpha} \tag{24}$$

- reduces to $\max R_s = 2\Delta f$ if $\alpha = 0$ (sinc)

Max. rate transmission with RC pulse: an example

- $\Delta f = 20$ MHz, 2-PAM, $\alpha = 1$
- max rate s.t. zero-ISI?

$$\max R_b = \max R_s = \frac{2\Delta f}{1+\alpha} = 20 \text{ [Mb/s]} \tag{25}$$

- and for wireless ? (BPSK)

$$\max R_b = \max R_s = \frac{\Delta f}{1+\alpha} = 10 \text{ [Mb/s]} \tag{26}$$

- compare to sinc pulse and make conclusions

RC pulse: an example

- R_b=1Mb/s, binary PAM (BPAM) or BPSK, RCP with $\alpha = 1$
- Bandwidth = ?
- Solution: BPAM

$$R_b = R_s = f_0 = 1 \text{ Msymb./s}$$

$$\Delta f_{BPAM} = \frac{1+\alpha}{2} f_0 = 1 \text{ MHz}$$

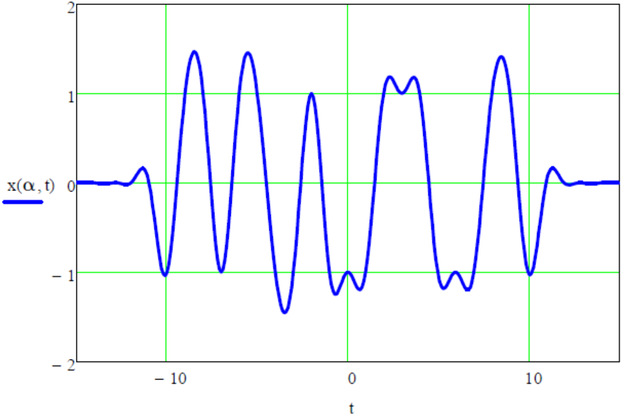
- Solution: BPSK

$$\Delta f_{BPSK} = 2\Delta f_{BPAM} = 2 \text{ MHz}$$

- 4-PAM, 4-PSK (QPSK): bandwidth = ?

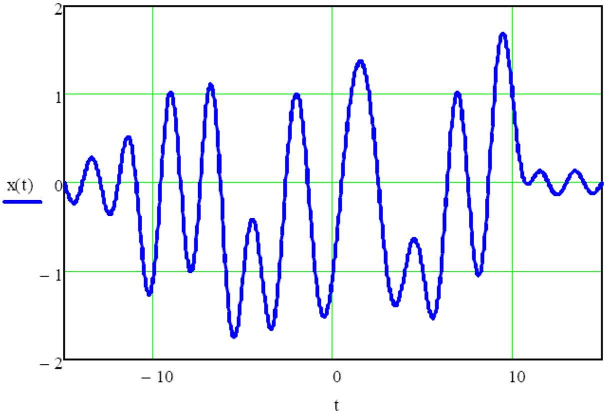
Sequential RC-pulse transmission of random data

- RC random signal $x(t) = \sum_{n=-10}^{10} a_n p_{rc}(t-n), \ a_n = \pm 1, \ \alpha = 1$



Sequential sinc-pulse transmission of random data

random data -> random signal $x(t) = \sum_{n=-10}^{10} a_n \mathrm{sinc}(t-n), \ \ a_n = \pm 1$



PSD of 2-PAM

- sequential 2-PAM transmission:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT_s), \ a_k = \pm A, \ \text{iid} \tag{27}$$

- truncate to $2N + 1$ pulses: $-N \leq k \leq N$

$$x_N(t) = \sum_{k=-N}^{N} a_k p(t - kT_s) \tag{28}$$

- and find its ESD via FT:

$$S_{xN}(f) = \mathrm{FT}\{x_N(t)\} = S_p(f) S_a(f), \tag{29}$$

- where

$$S_p(f) = \mathrm{FT}\{p(t)\}, \ S_a(f) = \sum_{k=-N}^{N} a_k e^{-j\omega k T_s} \tag{30}$$

PSD of 2-PAM

- so that the ESD is

$$E_{xN}(f) = \overline{|S_{xN}(f)|^2} = |S_p(f)|^2 \overline{|S_a(f)|^2} = (2N+1)A^2 |S_p(f)|^2 \quad (31)$$

- and therefore the PSD is

$$\boxed{P_x(f) = \lim_{N\to\infty} \frac{1}{(2N+1)T_s} E_{xN}(f) = \frac{A^2}{T_s}|S_p(f)|^2} \quad (32)$$

- where $|S_p(f)|^2$ is the (single) pulse ESD
- i.e.

$$\boxed{\text{PAM PSD} \sim \text{single pulse ESD}} \quad (33)$$

PSD of 2-PAM

- so that the ESD is

$$E_{xN}(f) = \overline{|S_{xN}(f)|^2} = |S_p(f)|^2 \overline{|S_a(f)|^2} = (2N+1)A^2 |S_p(f)|^2 \quad (34)$$

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$$P_x(f) = \lim_{N\to\infty} \frac{1}{(2N+1)T_s} E_{xN}(f) = \frac{A^2}{T_s} |S_p(f)|^2 \quad (35)$$

- this also holds for any (finite) N, not only $N \to \infty$!

$$\boxed{P_{xN}(f) = \frac{1}{(2N+1)T_s} E_{xN}(f) = \frac{A^2}{T_s} |S_p(f)|^2} \quad (36)$$

- average power (for any N):

$$\boxed{P_{xN} = \int_{-\infty}^{\infty} P_{xN}(f) df = \frac{A^2}{T_s} E_p} \quad (37)$$

PSD of 2-PAM: correlated data

- correlated $\{a_k\}$:

$$\overline{a_{k_1} a_{k_2}} = A^2 r_{\Delta k},\ \Delta k = k_1 - k_2,\ \overline{a_k} = 0 \tag{38}$$

- $r_{\Delta k}$ = normalized correlation
- the PSD is (for any N):

$$P_{xN}(f) = \frac{A^2}{T_s} |S_p(f)|^2 \left[1 + 2 \sum_{\Delta k=1}^{2N} \left(1 - \frac{\Delta k}{2N+1} \right) r_{\Delta k} \cos(\Delta k \omega T_s) \right] \tag{39}$$

- **Q1**: prove this!
- **Q2:** work out the special case of $r_{\Delta k} = 0\ \forall \Delta k > 1$ and $N \to \infty$.
- **Q3:** plot it for $p(t) = \mathrm{sinc}(t/T_s)$ and different values of r_1, e.g. $r_1 = 0, \pm 0.5, \pm 1$ and explain what you observe. What is the impact of data correlation on the PSD of 2-PAM?

PSD of 2-PAM: bandwidth

- show that, for any N,

$$\boxed{\Delta f_x = \Delta f_p} \tag{40}$$

- i.e. 2-PAM bandwidth = pulse bandwidth
- **Q:** what is the impact of data correlation on the bandwidth?

M-PAM

- how to increase R_b at the same bandwidth Δf?

M-PAM

- how to increase R_b at the same bandwidth Δf?
- multi-level PAM $\rightarrow$ M-PAM:

$$\{b_1, ..., b_{n_b}\} \ \rightarrow \ a \in [A_0, A_1 ... A_{M-1}], \ M = 2^{n_b} \tag{41}$$

- i.e. 1 symbols carrier n_b bits
- block bit mapping: $\{b_1, ..., b_{n_b}\} \ \rightarrow \ a$
- and sequential transmission

$$x(t) = \sum_{i=0}^{N-1} a_k p(t - kT_s) \tag{42}$$

M-PAM

- how to increase R_b at the same bandwidth Δf?
- multi-level PAM $\rightarrow$ M-PAM:

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$$x(t) = \sum_{i=0}^{N-1} a_k p(t - kT_s) \tag{42}$$

- $T_s \neq T_b$:

$$T_s = n_b T_b \ \rightarrow \ R_b = n_b R_s \tag{43}$$

- much higher rate if $n_b \gg 1$

M-PAM

- Nyquist criterion still applies
- i.e. sinc pulse maximizes data rate
- **Q:** find PSD and average power of M-PAM when all a_k are iid and

$$a_k \in [A_1, A_2 ... A_M],\ M = 2^{n_b},$$
$$A_m = A(2m - M - 1),\ m = 1...M \quad (44)$$

- where each a_k is uniformly distributed on $[A_1, A_2 ... A_M]$
- e.g. for $M = 4$:

$$M = 4:\ a_k \sim \text{uni}[-3A, -A, A, 3A] \quad (45)$$

Max. rate transmission with M-PAM

- $\Delta f = 20$ MHz, M-PAM, usually $M = 2^{n_b}$
- max rate s.t. zero-ISI?

$$\max R_b = \log M \cdot \max R_s = 2n_b \Delta f = 40 n_b \text{ [Mb/s]} \quad (46)$$

- and for wireless ? (BPSK)

$$\max R_b = n_b \max R_s = n_b \Delta f = 20 n_b \text{ [Mb/s]} \quad (47)$$

- max-rate pulse $p(t) = \text{sinc}(t/T_s)$

Summary

- how to transmit 1 bit?
- pulse-amplitude modulation
- bandlimited channels and ISI
- Nyquist zero-ISI criterion
- sinc and RC pulses
- max. rate over bandlimited channels
- PSD of PAM, bandwidth
- M-PAM

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