ELG5375: Digital Communications

Lecture 3

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Review of Signals, Systems & Fourier Analysis

- Fourier Transform/Series
  - important tool
  - time domain $\leftrightarrow$ frequency domain
  - simplifies analysis
  - key system parameter: bandwidth
- studied well at undergrad level
- only brief review here
- consult books/notes for more info[1,2]

[1] R.E. Ziemer, W.H. Tranter, Principles of Communications, Wiley, 2009.

[2] B.P.Lathi, Z. Ding, Modern Digital and Analog Communication Systems, Oxford University Press, 2009.

Fourier Transform (FT)

- decompose signal into complex exponents (or sin/cos)

$$x(t) \leftrightarrow S(f) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \ \omega = 2\pi f \tag{1}$$

$$x(t) = \int_{-\infty}^{\infty} S(f) e^{j\omega t} df \tag{2}$$

- $S(f) =$ FT or (double-sided) spectrum of $x(t)$
- $f =$ (linear) frequency [Hz], $\omega =$ radial frequency [rad./s]
- includes both positive and negative frequencies

Fourier Transform (FT)

- if $x(t)$ is real-valued: $\text{Im}\{x(t)\} = 0$ or $x(t) = x^*(t)$,

$$S(-f) = S^*(f) \tag{3}$$

- only positive frequencies suffice:

$$x(t) = \int_0^\infty |S(f)| \cos(\omega t + \theta(f)) df, \ \theta(f) = \arg\{S(f)\} \tag{4}$$

- single-sided spectrum: $S(f) \ \forall \ f \geq 0$

Examples

- rectangular pulse $\Pi(t/T)$:

$$x(t) = \Pi(t/T) \triangleq \begin{cases} 1, & \text{if } |t| \leq T/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\leftrightarrow \; S(f) = T\text{sinc}(Tf), \; sinc(f) \triangleq \frac{\sin(\pi f)}{\pi f} \quad (5)$$

- triangular pulse $x(t) = \Lambda(t/T)$:

$$\Lambda(t/T) \triangleq \begin{cases} 1 - |t|/T, & \text{if } |t| \leq T \\ 0, & \text{otherwise} \end{cases} \quad \leftrightarrow \; S(f) = T\text{sinc}^2(Tf) \quad (6)$$

- sinc pulse $x(t) = \text{sinc}(t/T)$:

$$x(t) = \text{sinc}(t/T) \leftrightarrow \; S(f) = T\Pi(Tf)$$

Important Properties[3,4]

- linearity, time shift, modulation, duality
- convolution

$$y(t) = x_1(t) * x_2(t) \triangleq \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$
$$\leftrightarrow \;\; S_y(f) = S_1(f) S_2(f) \tag{7}$$

- Parseval identity:

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = \int_{-\infty}^{\infty} S_1(f) S_2^*(f) df \tag{8}$$

- i.e. scalar products in time and frequency domains are the same

[3] R.E. Ziemer, W.H. Tranter, Principles of Communications, Wiley, 2009.

[4] B.P.Lathi, Z. Ding, Modern Digital and Analog Communication Systems, Oxford University Press, 2009.

Important Properties

- Rayleigh energy theorem (for energy-type signals):

$$E_x \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |S(f)|^2 df \tag{9}$$

- i.e., energy in time domain = energy in frequency domain
- $x(t)$ = normalized voltage/current (in 1 Ohm resistor)
- E_x = energy dissipated in 1 Ohm resistor

Bandwidth

- defined for positive frequencies only (no negative frequencies in real physical world)
- informally: **smallest range of frequencies that contains all or most of signal's energy**
- absolute bandwidth: contains all signal energy
- for baseband signals: minimum Δf such that

$$|S(f)| = 0 \ \forall f > \Delta f \tag{10}$$

- for bandpass/RF signals: $\Delta f \triangleq \min(f_2 - f_1)$ such that

$$|S(f)| = 0 \ \forall f \notin [f_1, f_2], \ 0 \leq f_1 \leq f_2 \tag{11}$$

Examples

- sinc pulse $x(t) = \text{sinc}(t/T)$:

$$S(f) = T\Pi(Tf) \;\rightarrow\; \Delta f = \frac{1}{2T} \tag{12}$$

- i.e. shrinking pulse $\rightarrow$ increasing bandwidth (general property)
- rectangular pulse $x(t) = \Pi(t/T)$:

$$S(f) = T\text{sinc}(Tf) \;\rightarrow\; \Delta f = \infty \tag{13}$$

- use 1st null bandwidth Δf_0 (contains most energy):

$$\Delta f_0 \triangleq \min\; \Delta f :\; S(\Delta f) = 0 \;\rightarrow\; \Delta f_0 = 1/T \tag{14}$$

- **Q**: find the bandwidth of (i) triangular pulse, (ii) $\text{sinc}^2(t/T)$, and (iii) $\cos^2(\omega_0 t)$

Impulse Response

- simple way to find the output of linear time-invariant (LTI) system (e.g. a filter)

$$x(t) \;\rightarrow\; \text{LTI} \;\rightarrow\; y(t) = x(t) * h(t) \triangleq \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad (15)$$

- $h(t)$ = impulse response:

$$\delta(t) \;\rightarrow\; \text{LTI} \;\rightarrow\; y(t) = h(t) \quad (16)$$

- i.e impulse response is the response of LTI system to delta function at its input

Delta function

- $\delta(t)$ = Dirack delta-function (generalized function or distribution)
- defined by action, not values!

$$\forall x(t): \; x(t) * \delta(t) = x(t) \; \rightarrow \; \int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0) \tag{17}$$

- properties:

$$\begin{aligned} &\delta(t) \geq 0, \; \delta(t) = 0 \; \forall \; t \neq 0, \; x(t)\delta(t) = x(0)\delta(t) \\ &\int_{-\infty}^{\infty} \delta(t)dt = 1, \end{aligned} \tag{18}$$

- linear operations are OK, but non-linear - illegal! (e.g. cannot square)

Frequency Response

- frequency response $H(f)$ = response of LTI system to $e^{j\omega t}$

$$e^{j\omega t} \;\rightarrow\; \text{LTI} \;\rightarrow\; y(t) = H(\omega)e^{j\omega t} \tag{19}$$

- i.e $e^{j\omega t}$ is an eigenfunction of LTI system
- important properties:

$$H(f) = \mathrm{FT}\{h(t)\} \tag{20}$$

$$y(t) = x(t) * h(t) \;\leftrightarrow\; S_y(f) = H(f)S_x(f) \tag{21}$$

- greatly simplifies analysis!
- LTI systems are best analysed in frequency domain, nonlinear - in time domain

Periodic Signals & Fourier Series

- periodic signal:

$$\forall t: \ x(t) = x(t+T), \ T > 0 \tag{22}$$

- T = period
- Fourier series (FS):

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \ c_n = \frac{1}{T}\int_T x(t) e^{-jn\omega_0 t} dt, \tag{23}$$

- $\omega_0 = 2\pi f_0, \ f_0 = 1/T$ = fundamental frequency [Hz]

LTI Response to Periodic Signals

- $x(t) =$ periodic, at the input:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \rightarrow \text{LTI} \rightarrow y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) c_n e^{jn\omega_0 t} \tag{24}$$

- i.e the output $y(t)$ is also periodic (same fundamental frequency) and

$$\{c_n\} \rightarrow \text{LTI} \rightarrow \{H(n\omega_0)c_n\} \tag{25}$$

- the input-output relationship in frequency domain

Energy and Power

- important resource in communications
- battery-operated devices: very limited (small) energy
- wireless communications: interference to other systems/users (difficult to design)
- limited-power amplifiers
- cost (and availability) of electricity
- energy/power efficiency is a key performance indicator

Energy/Power Type Signals

- **energy-type signals**: finite energy

$$E_x \triangleq \int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty \qquad (26)$$

- **power-type signals**: finite, non-zero power

$$P_x \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt < \infty, \;\; P_x > 0 \qquad (27)$$

  i.e. power is (average) energy per time.

- cannot be both: $P_x > 0$ implies $E_x = \infty$ (prove it!)
- likewise, $E_x < \infty$ implies $P_x = 0$
- $x(t) =$ normalized voltage/current (in 1 Ohm resistor)
- $E_x, P_x =$ energy, power dissipated in 1 Ohm resistor

Energy/Power Type Signals

- in the physical world/engineering, $E_x < \infty$ (so that $P_x = 0$)
- yet, power-type signals are useful model
- practical power is defined for large but finite T:

$$P_{x,T} \triangleq \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt \tag{28}$$

so that $P_{x,T} \approx P_x$.

- if $x(t)$ is periodic with period T_0,

$$P_x = P_{x,T_0} = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt \tag{29}$$

i.e. one can limit integration to 1 (or more) period(s) and all equalities are *exact*.

Examples

- sinusoidal signal

$$x(t) = A\cos(\omega_c t + \theta) \;\rightarrow\; P_x = A^2/2,\; E_x = \infty \tag{30}$$

- rectangular pulse (of duration T)

$$x(t) = A\Pi(t/T) \;\rightarrow\; P_x = 0,\; E_x = A^2 T \tag{31}$$

- truncated sinusoid (of duration T):

$$x(t) = A\Pi(t/T)\cos(\omega_c t) \;\rightarrow\; P_x = 0,\; E_x = A^2 T/2 \tag{32}$$

Energy Spectral Density (ESD)

Defined for energy-type signals. Tells us how the energy is distributed across different frequencies.

Defining properties:

1. non–negative, measured in [J/Hz]:

$$E_x(f) \geq 0 \tag{33}$$

2. for any interval $[f_1, f_2]$, its energy content E_{12} is

$$E_{12} = \int_{f_1}^{f_2} E_x(f) df \tag{34}$$

Energy Spectral Density (ESD)

#2 implies that ESD integrates to the total energy E_x:

$$E_x \triangleq \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} E_x(f) df \tag{35}$$

and that the energy content ΔE_x of small interval Δf around f_0 is

$$\Delta E_x \approx E_x(f_0) \Delta f \tag{36}$$

justifying the term "energy density".

Energy Spectral Density (ESD)

ESD via FT:

$$E_x(f) = |S_x(f)|^2 \text{ [J/Hz]},\ S_x(f) \triangleq \text{FT}\{x(t)\} \tag{37}$$

Input-output relationship:

If $y(t) = h(t) * x(t)$ is LTI filter's output, its ESD $E_y(f)$ is

$$E_x(f) \ \rightarrow \ \text{LTI} \ \rightarrow \ E_y(f) = |H(f)|^2 E_x(f) \tag{38}$$

and its total energy

$$E_y \triangleq \int_{-\infty}^{+\infty} |y(t)|^2 dt = \int_{-\infty}^{+\infty} |H(f)|^2 E_x(f) df \tag{39}$$

Power Spectral Density (PSD)

Defined for power-type signals. Similar to ESD, using "power is energy per time" principle.

Defining properties:

1. non–negative, measured in [W/Hz]:

$$P_x(f) \geq 0 \tag{40}$$

2. for any interval $[f_1, f_2]$, its power content P_{12} is

$$P_{12} = \int_{f_1}^{f_2} P_x(f) df \tag{41}$$

Power Spectral Density (PSD)

#2 implies that PSD integrates to the total power P_x:

$$P_x \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} P_x(f) df \tag{42}$$

and that the power content ΔP_x of small interval Δf around f_0 is

$$\Delta P_x \approx P_x(f_0)\Delta f \tag{43}$$

justifying the term "power density".

Power Spectral Density (PSD)

Let $x_T(t)$ be $x(t)$ truncated to $[-T, T]$:

$$x_T(t) = \begin{cases} x(t), \text{ if } |t| \leq T \\ 0, \text{ otherwise} \end{cases} \quad (44)$$

and $E_{x_T}(f) = |S_{x_T}(f)|^2$ be its ESD, $S_{x_T}(f) \triangleq \text{FT}\{x_T(t)\}$.

Then, it follows that

$$P_x(f) = \lim_{T\to\infty} \frac{1}{2T} E_{x_T}(f) = \lim_{T\to\infty} \frac{1}{2T} |S_{x_T}(f)|^2 \ \ \text{[W/Hz]} \quad (45)$$

Eq. (45): complies with "power is energy per time" principle, i.e. "PSD is ESD per time".

Warning: $P_x(f) \neq |S_x(f)|^2$!

Power Spectral Density (PSD)

Input-output relationship:

If $y(t) = h(t) * x(t)$ is LTI filter's output, its PSD $P_y(f)$ is

$$P_x(f) \ \rightarrow \ \text{LTI} \ \rightarrow \ P_y(f) = |H(f)|^2 P_x(f) \tag{46}$$

and its total power

$$P_y = \int_{-\infty}^{+\infty} |H(f)|^2 P_x(f) df \tag{47}$$

$|H(f)|^2 =$ power gain.

Importance of PSD/ESD

- determines signal's bandwidth
- determines out of band emissions (interference)
- both must comply with regulations! (enforced)
- determines total power/energy and thus power/energy efficiency
- one of key characteristics for design

Sampling Theorem

- foundation for all digital systems
- bridge between analog (physical) and digital (Internet) worlds
- analog-to-digital conversion (ADC) or pulse-code modulation (PCM)
- informally: band-limited signal can be completely recovered from its samples
- i.e. samples carry all the information about signal
- sampling frequency: at least twice the bandwidth

The Sampling Theorem

A continuous, bandlimited signal $x(t)$ with absolutely-integrable FT can be completely recovered from its samples $\{x(nT_s)\}$:

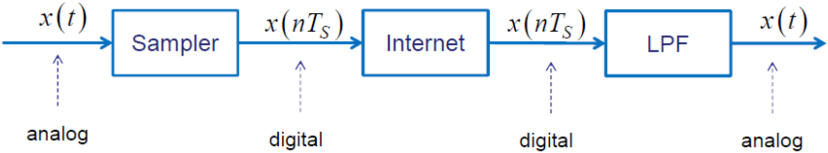
$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}\left(\frac{t}{T_s} - n\right) \quad (48)$$

- T_s is the sampling interval, $T_s = 1/f_s$
- f_s is the sampling frequency, $f_s \geq 2\Delta f$
- Δf is the signal's (absolute) bandwidth

i.e., knowing $\{x(nT_s)\} \Leftrightarrow$ knowing $x(t)$

Sampling Theorem and The Internet

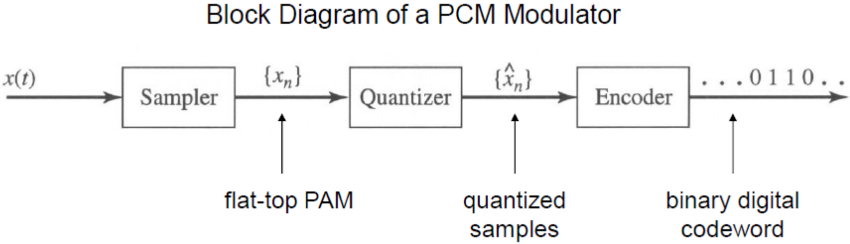
(somewhat simplified: no quantizing yet)



$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_S)\,\mathrm{sinc}\left(\frac{t}{T_S} - n\right)$$

Analog-to-Digital Conversion (ADC)

- also known as "pulse-code modulation" (PCM)
- key idea: 1st sample $x(t)$ then quantize $\{x(nT_s)\}$
- **quantizing**: $\{x(nT_s)\}$ is rounded off to the closest allowed level (only a finite number of levels are used)
- extensive applications: digital audio and video (.mp3, .mp4, etc)



Summary

- signals & systems
- Fourier transform & series
- properties & use
- bandwidth
- impulse/frequency response
- ESD/PSD
- input-output relationship
- sampling theorem, ADC/PCM

Reading

- R.E. Ziemer, W.H. Tranter, Principles of Communications, Wiley, 2009.
- B.P.Lathi, Z. Ding, Modern Digital and Analog Communication Systems, Oxford University Press, 2009.
- S. Haykin, Digital Communication Systems, Wiley, 2014.