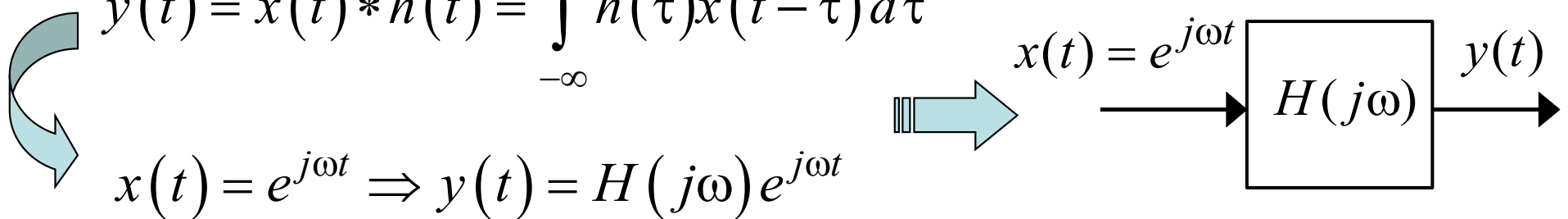


Review of Fourier Series

- Why Fourier series?
- Complex exponents are the eigenfunctions of LTI systems,



$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$

$$x(t) = e^{j\omega t} \Rightarrow y(t) = H(j\omega) e^{j\omega t}$$

The diagram illustrates the relationship between convolution and frequency response. A curved arrow points from the convolution equation to the eigenfunction equation. A block arrow points from the eigenfunction equation to a block diagram of an LTI system. The block diagram shows an input signal $x(t) = e^{j\omega t}$ entering a block labeled $H(j\omega)$, which outputs $y(t)$.

- $x(t)$ is presented as a linear combination of complex exponents (with different frequencies)
- LTI system response is the same linear combination of individual responses.

Periodic Signals

- Periodic signals are very important class of signals (widely used), where smallest T is a period,

$$x(t) = x(t + T), \text{ for all } t$$

- Examples: $\cos(\omega_0 t)$ & $e^{j\omega_0 t}$. Period $T = 2\pi / \omega_0$
- Introduce a set of harmonically-related complex exponents,

$$\phi_n(t) = e^{jn\omega_0 t} = e^{jn\frac{2\pi}{T}t}, \quad n = 0, \pm 1, \pm 2, \dots$$

DC
1st harmonic

- Construct a periodic signal,

$$x'(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

Fourier Series of Periodic Signal

- Can $x'(t)$ be made the same as $x(t)$?
- Yes, by adjusting c_n ,

$$c_n = \frac{1}{T} \int_T x(t) e^{-j2\pi \frac{n}{T} t} dt \quad \Rightarrow \quad x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}, \omega_0 = \frac{2\pi}{T}$$

- $\{c_n\}$ – Fourier series coefficients (or spectral coefficients, or discrete spectrum of the signal)
- c_0 – DC component or average value of $x(t)$,

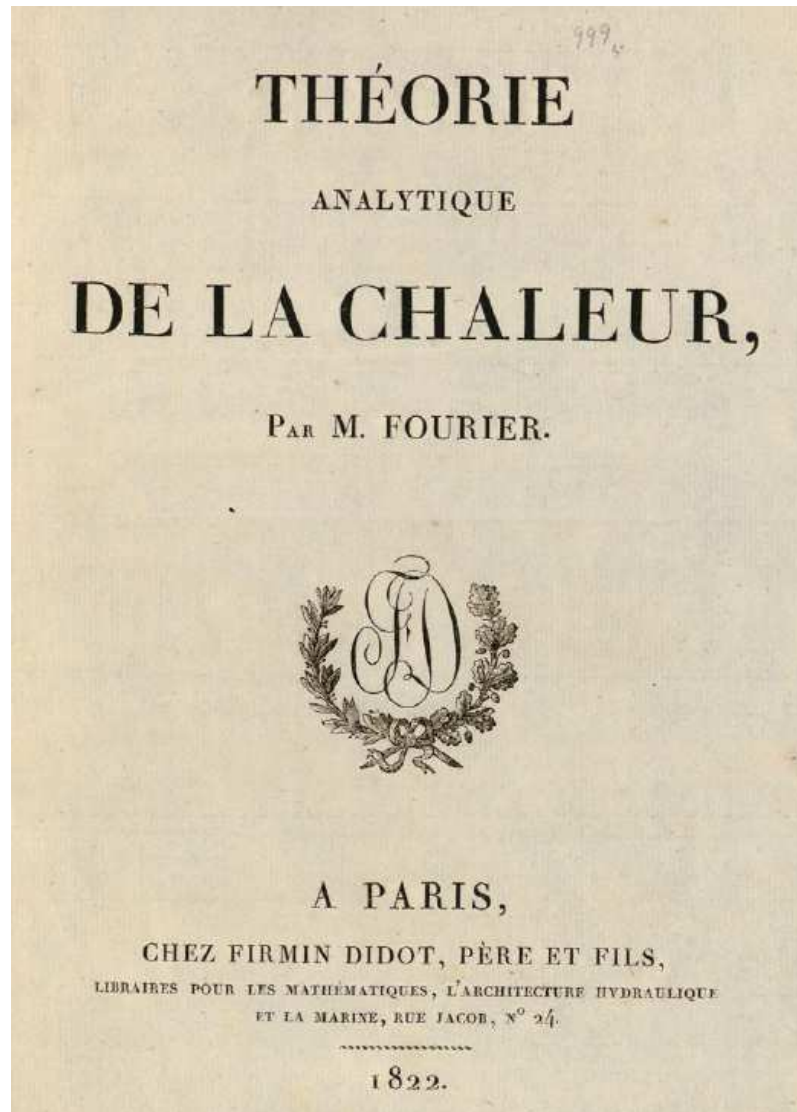
$$c_0 = \frac{1}{T} \int_T x(t) dt$$

Joseph Fourier

- **Born:** 21 Mar. 1768, Auxerre, Burgundy, France
- **Died:** 16 May 1830 (aged 62), Paris, France
- Fourier series/transform
- Fourier's law of heat conduction
- Fourier–Motzkin elimination (of variables in a system of linear inequalities)
- Greenhouse effect (a.k.a. “global warming”)
- https://en.wikipedia.org/wiki/Joseph_Fourier
- https://en.wikipedia.org/wiki/Fourier_series

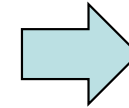


Fourier Series, 1807-1822



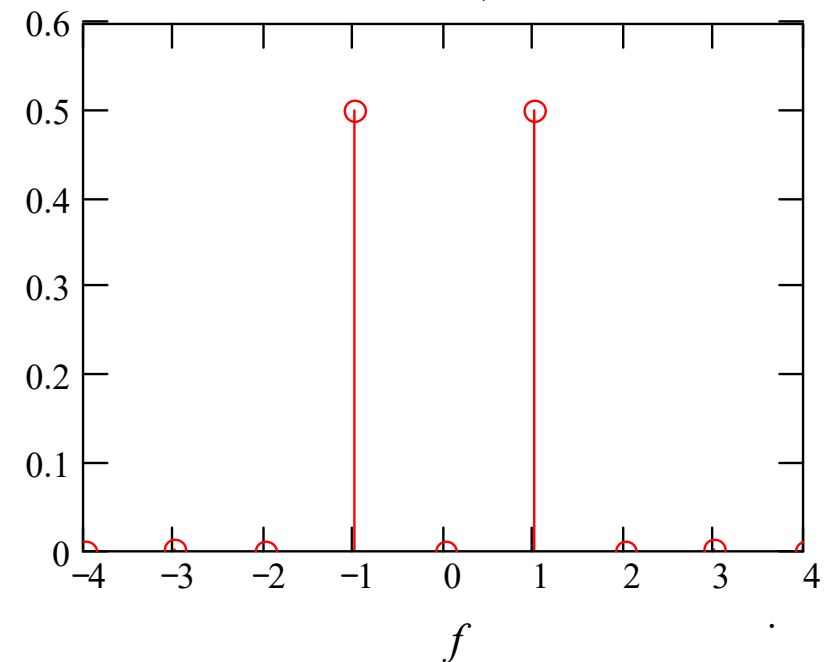
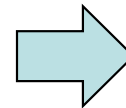
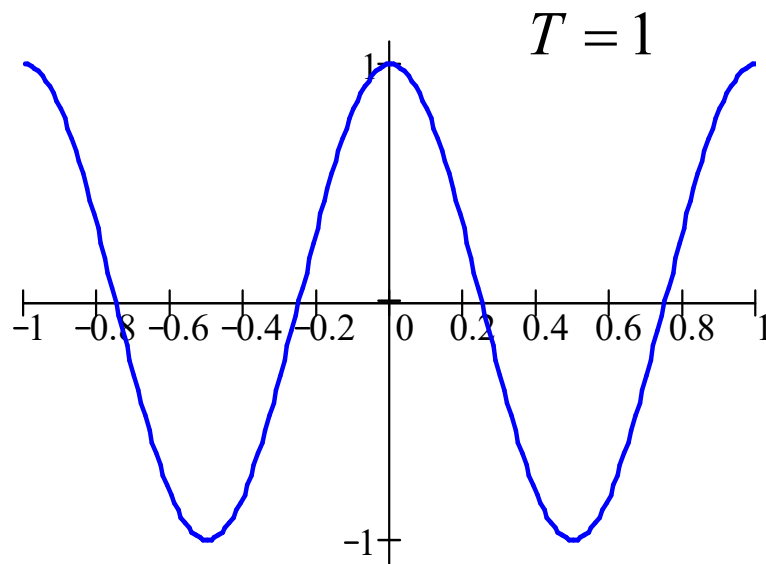
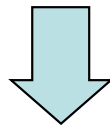
Example of Fourier Series

$$x(t) = \cos(\omega_0 t) = \frac{1}{2} \left(e^{j\omega_0 t} + e^{-j\omega_0 t} \right)$$



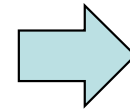
$$c_1 = \frac{1}{2}, c_{-1} = \frac{1}{2},$$

$$c_k = 0, k \neq \pm 1$$



Example of Fourier Series

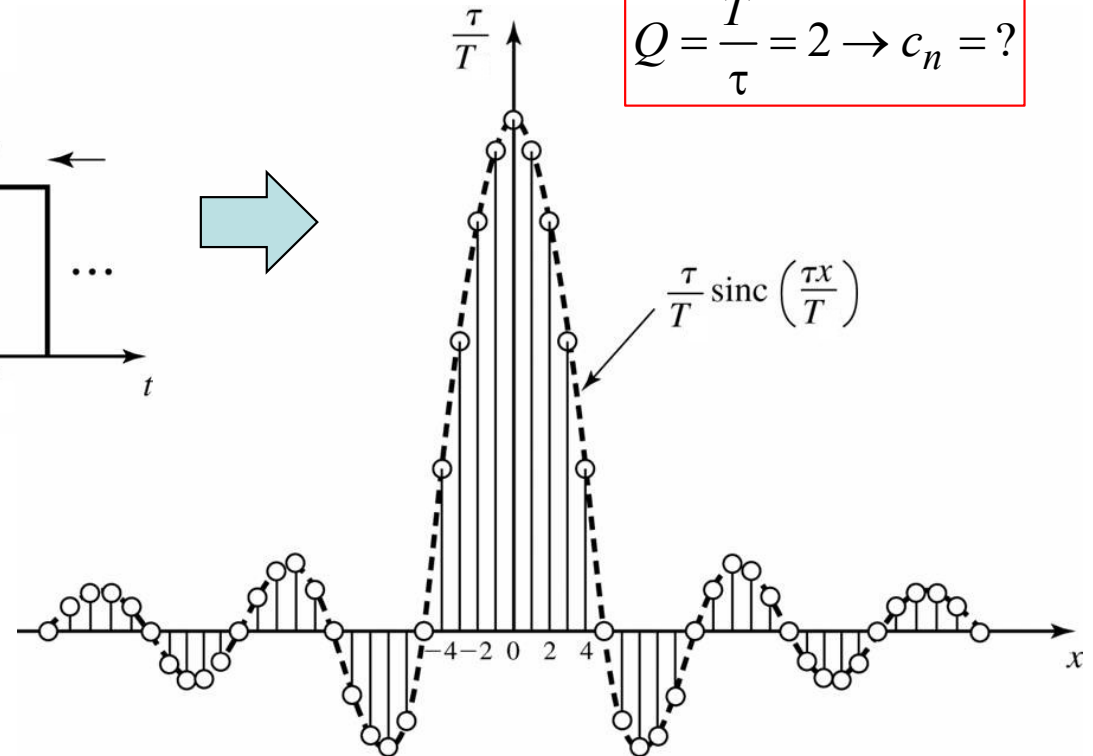
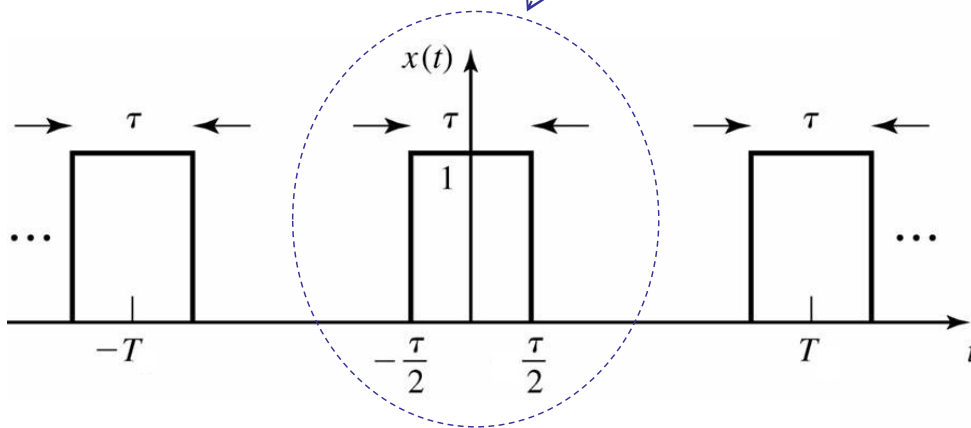
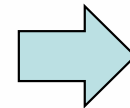
$$x(t) = \begin{cases} 1, & |t| < \tau/2 \\ 0, & \tau/2 < |t| < T/2 \end{cases}$$



$$c_n = \frac{\tau}{T} \operatorname{sinc}\left(\frac{n\tau}{T}\right),$$

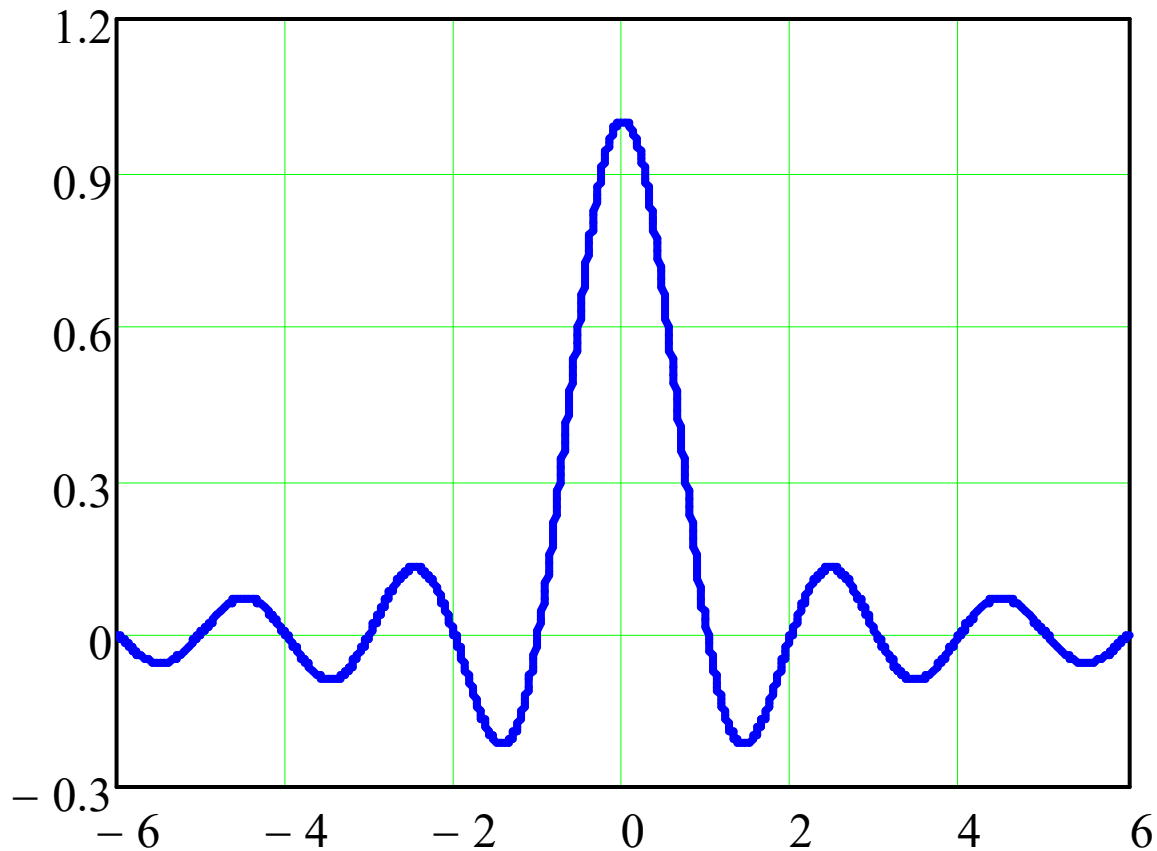
$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

$$Q = \frac{T}{\tau} = 2 \rightarrow c_n = ?$$



Q.: How does c_n scale with the pulse amplitude?
Duration? Period?

$\text{sinc}(t)$

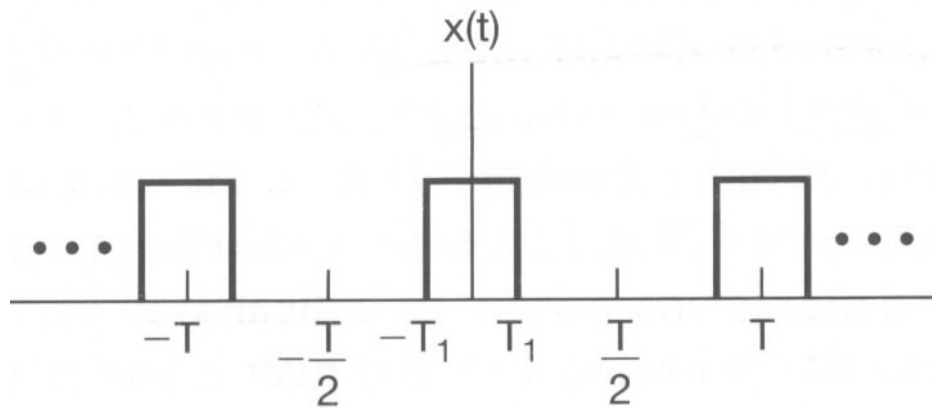


$$\text{sinc}(n) = 0, \quad n \neq 0$$

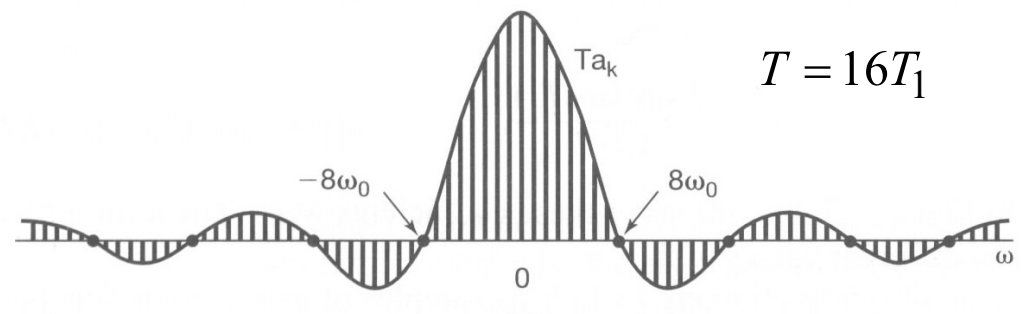
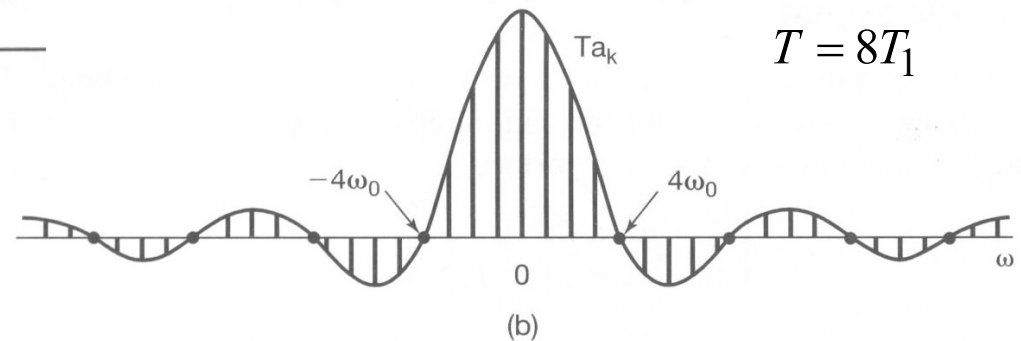
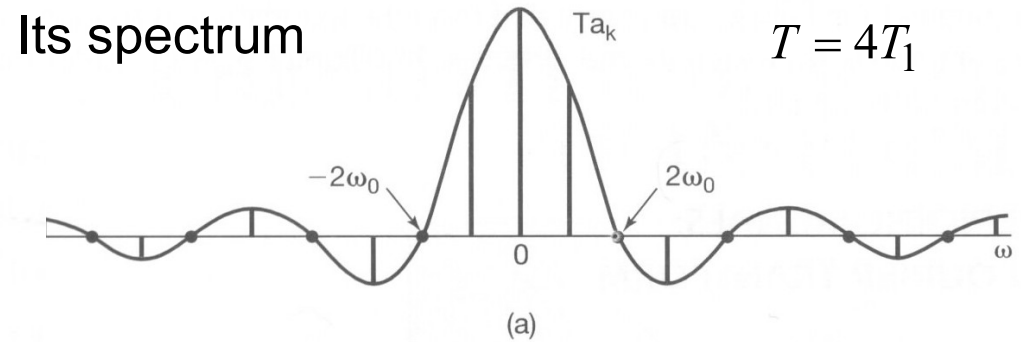
$$\text{sinc}(0) = ?$$

Example of Fourier Series

Periodic signal



Duty cycle: $Q = \frac{T}{\tau} = \frac{T}{2T_1}$



A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

Q.: How does c_n scale with
the pulse amplitude?
Duration? Period?

Convergence of Fourier Series

- Dirichlet conditions:
 - $x(t)$ must be absolutely integrable (finite power)

$$\int_T |x(t)| dt < \infty$$

- $x(t)$ must be of bounded variation; that is the number of maxima and minima during a period is finite
 - In any finite interval of time, there are only a finite number of discontinuities, which are finite.
- Dirichlet conditions are only sufficient, but are not necessary.
- All physical (practical) periodic signals do have Fourier series.
- **Q: show that 1st condition holds if $x(t)$ is bounded**

Fourier Series of Real Signals

- For a real signal, $\text{Im}\{x(t)\} = 0 \Rightarrow c_{-n} = c_n^*$
- The trigonometric (sin/cos) Fourier series:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)], \quad \omega_0 = \frac{2\pi}{T}$$

$$a_n = 2 \text{Re}\{c_n\} = \frac{2}{T} \int_T x(t) \cos(n\omega_0 t) dt, \quad b_n = -2 \text{Im}\{c_n\} = \frac{2}{T} \int_T x(t) \sin(n\omega_0 t) dt$$

- Another form (cos only):

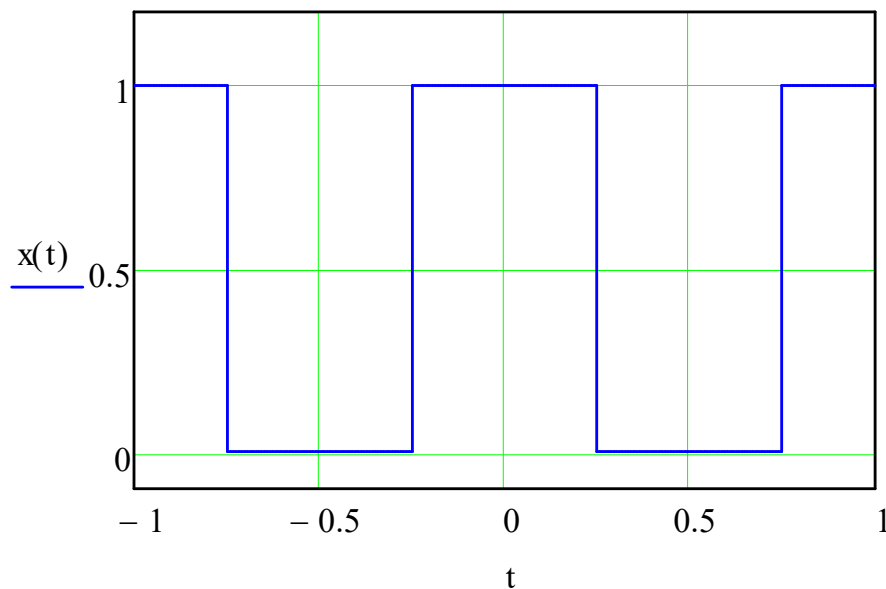
$$x(t) = x_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \varphi_n)$$

$$A_n = 2|c_n| = \sqrt{a_n^2 + b_n^2}, \quad \varphi_n = -\arg(c_n) = -\tan^{-1}(b_n / a_n)$$

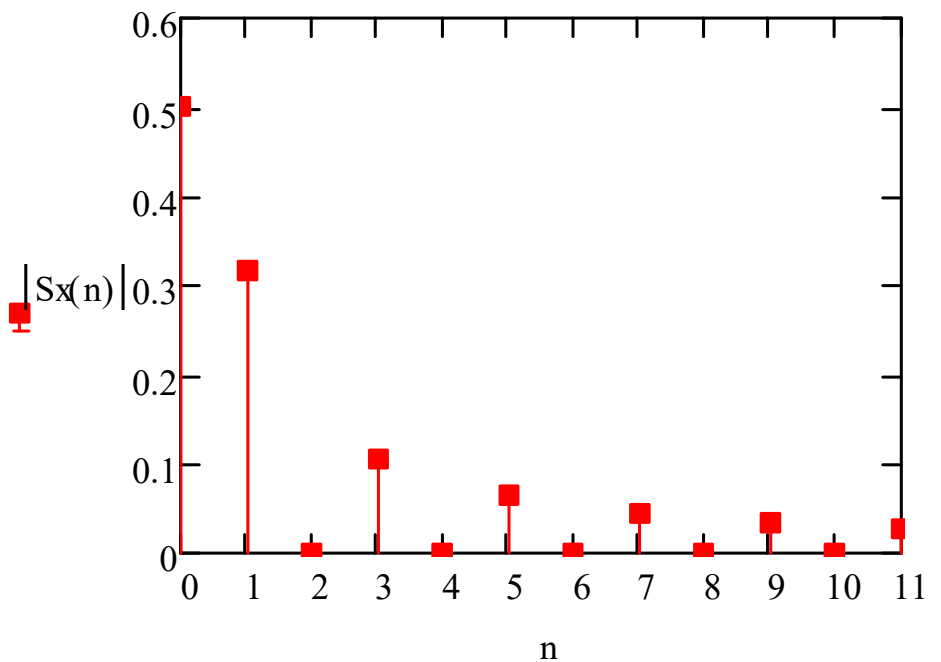
- Single-sided spectrum: only positive frequencies, on SA

Example: rectangular pulse train

Signal (OS, time domain)



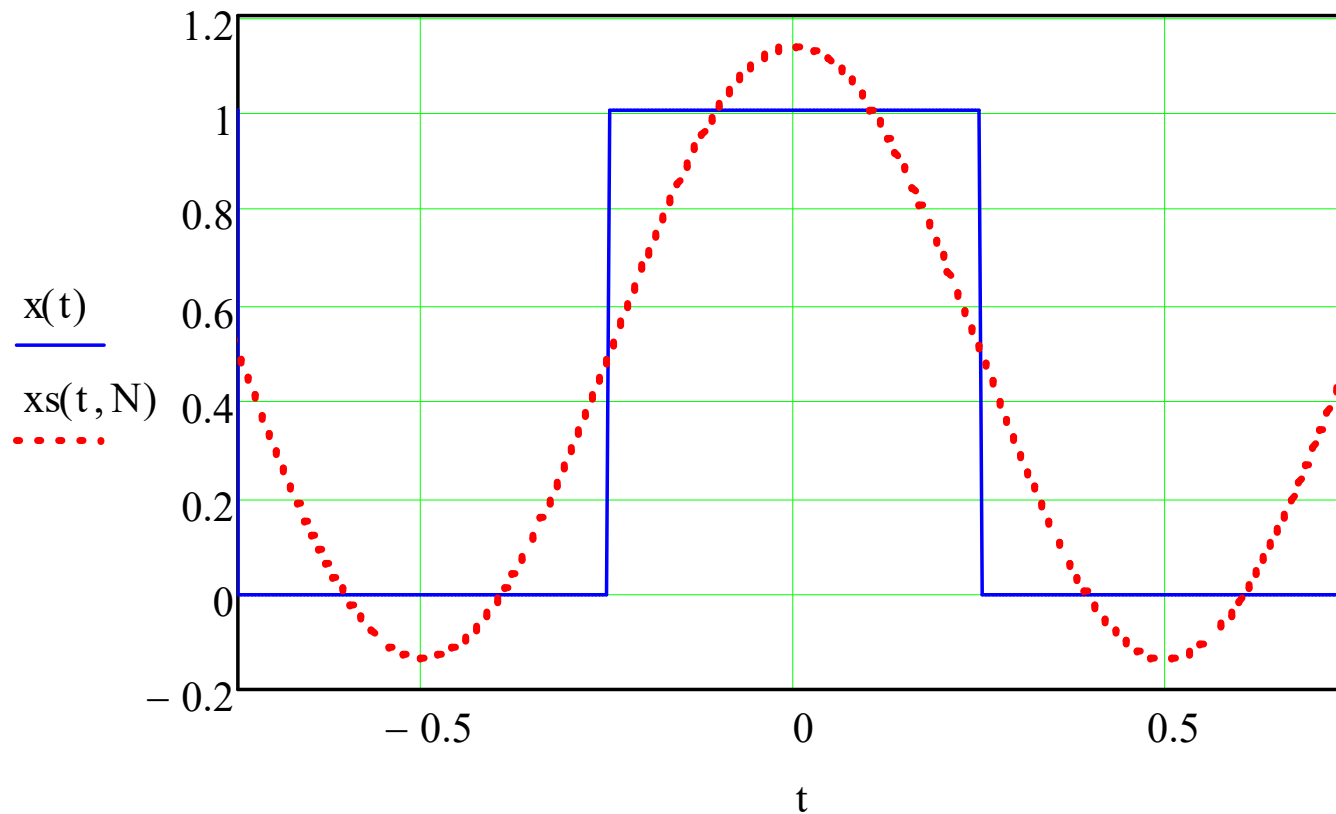
Spectrum (SA, frequency domain)



$$x_s(t, N) := \frac{1}{2} + \sum_{n=1}^N \left(\text{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

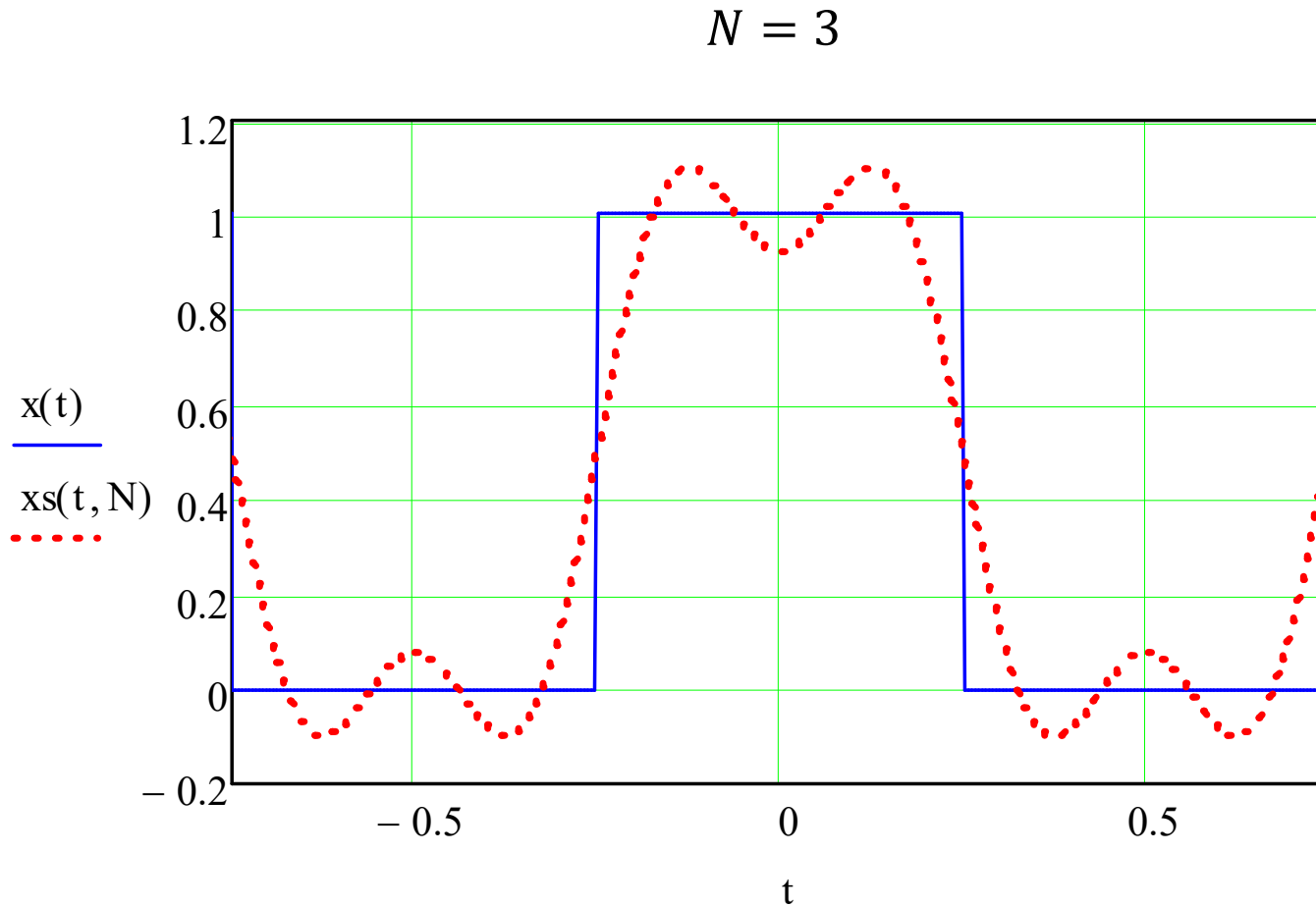
Convergence of Fourier Series

$N = 1$



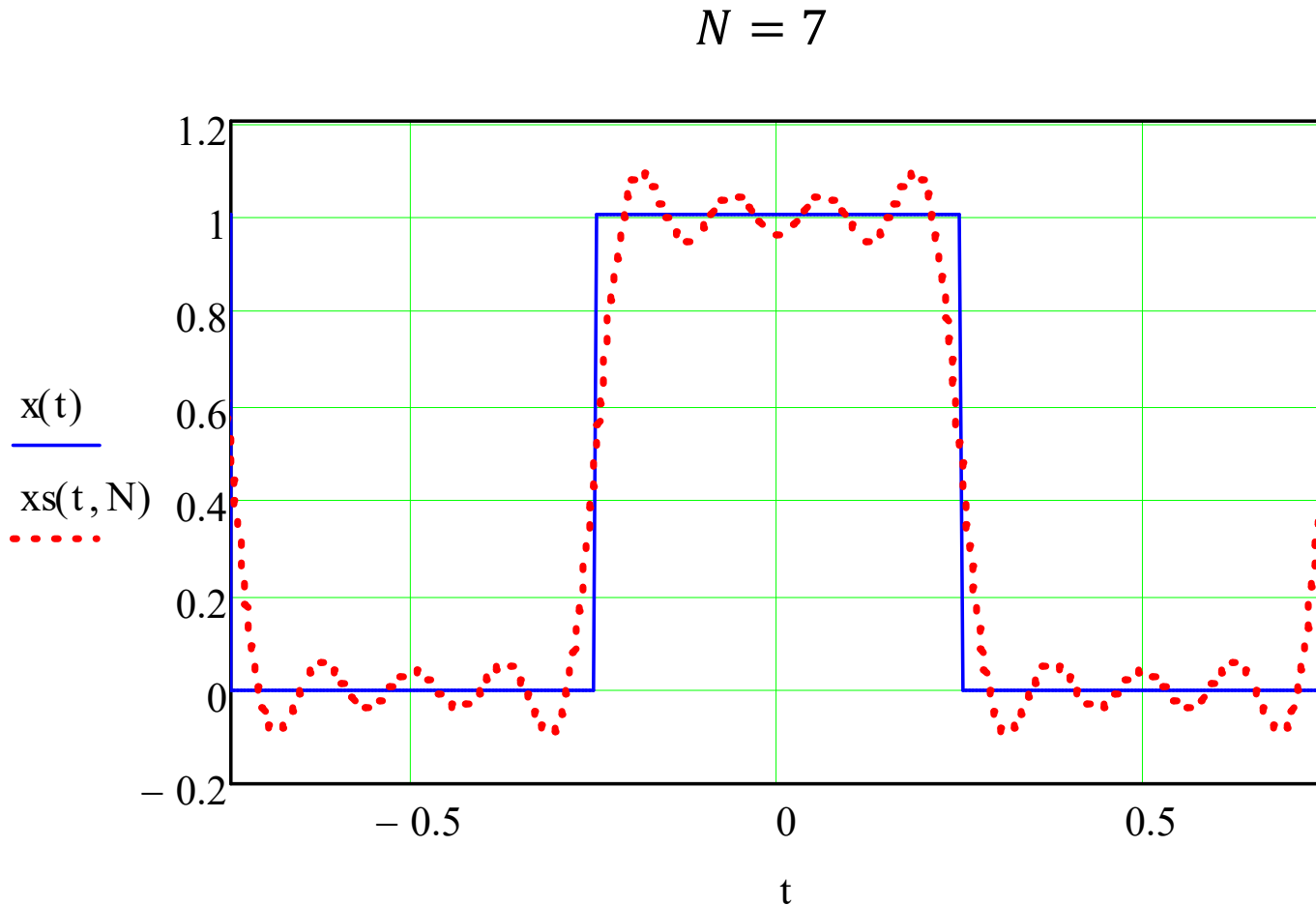
$$x_s(t, N) := \frac{1}{2} + \sum_{n=1}^N \left(\text{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

Convergence of Fourier Series



$$x_s(t, N) := \frac{1}{2} + \sum_{n=1}^N \left(\operatorname{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

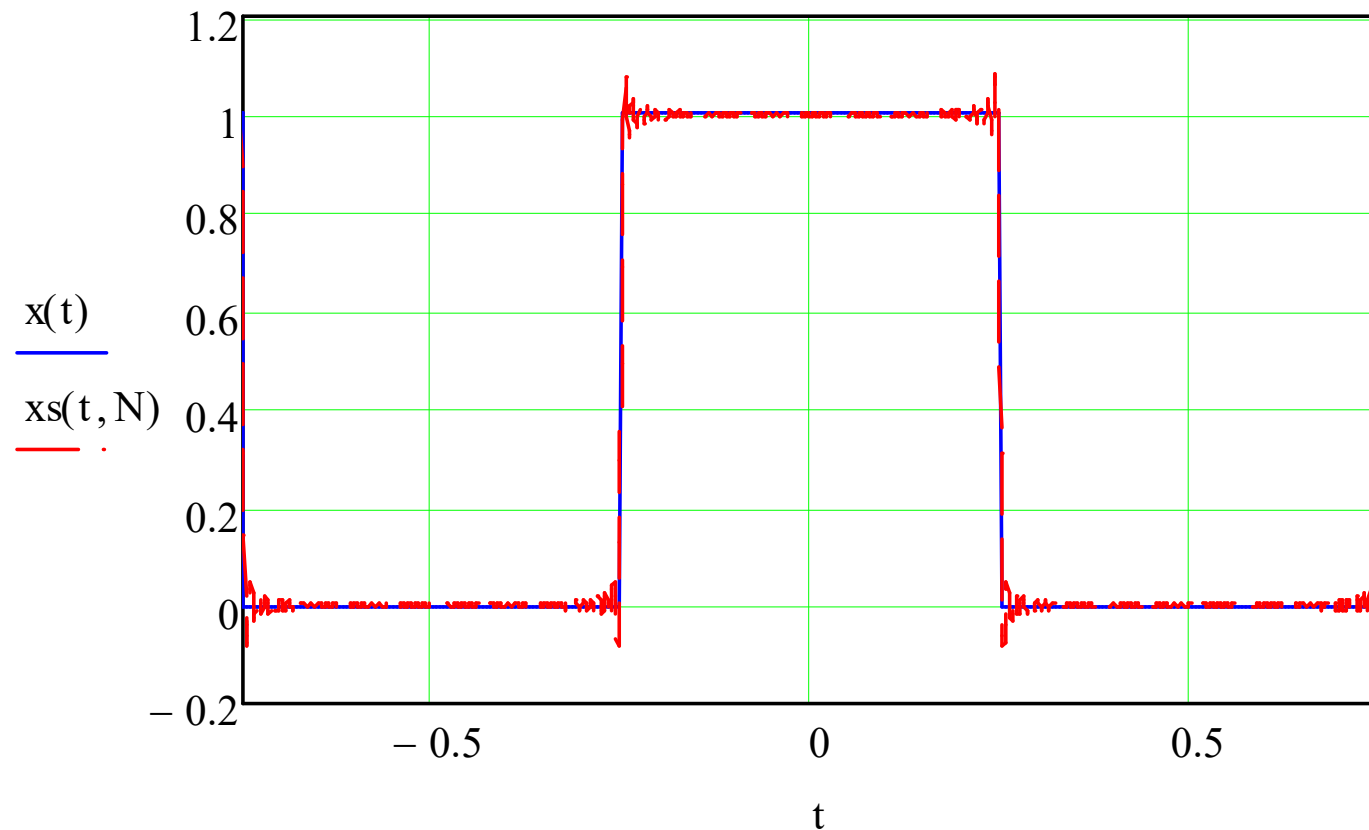
Convergence of Fourier Series



$$x_s(t, N) := \frac{1}{2} + \sum_{n=1}^N \left(\operatorname{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

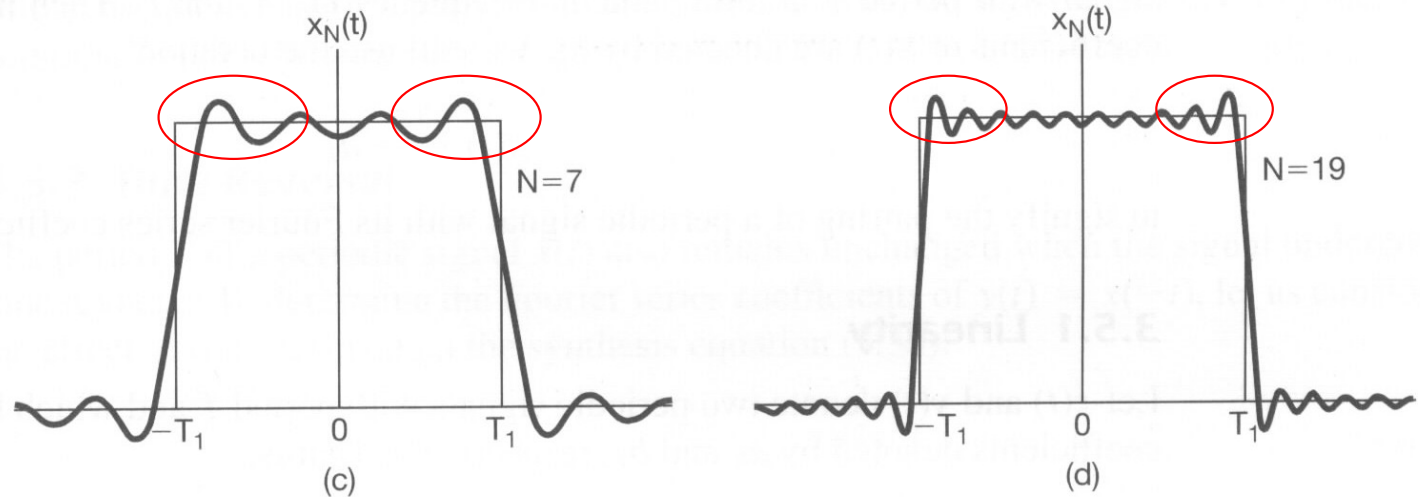
Convergence of Fourier Series

$N = 100$



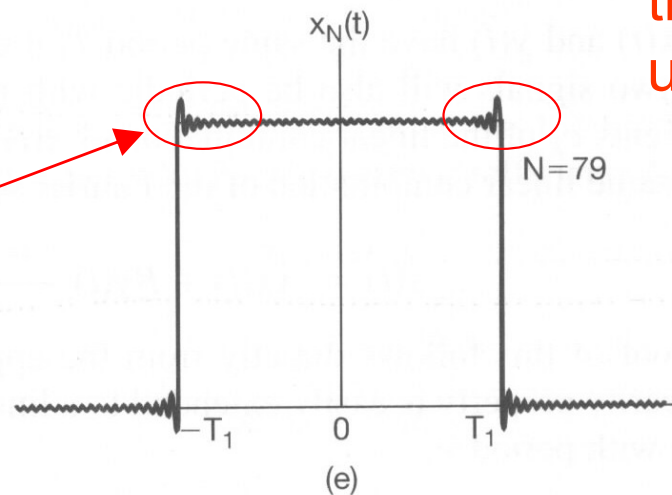
$$x_s(t, N) := \frac{1}{2} + \sum_{n=1}^N \left(\text{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

Gibbs Phenomenon



Q.: reproduce these graphs using a computer

increasing the number of terms does not decrease the ripple maximum!



Properties of Fourier Series

- Linearity:
$$F [\alpha x_1(t) + \beta x_2(t)] = \alpha F [x_1(t)] + \beta F [x_2(t)]$$
- Time shifting:
$$x(t) \xleftrightarrow{F} c_n \Leftrightarrow x(t - t_0) \xleftrightarrow{F} e^{-jn\omega_0 t_0} c_n$$
- Time reversal:
$$x(t) \xleftrightarrow{F} c_n \Leftrightarrow x(-t) \xleftrightarrow{F} c_{-n}$$
- Time scaling:
$$x(\alpha t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn(\alpha\omega_0)t}$$

Q.: prove these properties

Properties of Fourier Series

- Multiplication:

$$x(t)y(t) \xleftrightarrow{F} \sum_{k=-\infty}^{\infty} c'_k c''_{n-k}$$

- Convolution:

$$\int_T x(\tau)y(t-\tau)d\tau \xleftrightarrow{F} Tc'_n c''_n$$

- Differentiation:

$$\frac{dx(t)}{dt} \xleftrightarrow{F} jn\omega_0 c_n$$

- Integration:

$$\int_{-\infty}^t x(\tau)d\tau \xleftrightarrow{F} \frac{c_n}{jn\omega_0}, \quad \text{for } c_0 = 0$$

Q.: prove these properties

Properties of Fourier Series

- Real $x(t)$:

$$c_{-n} = c_n^*$$

- Real & even $x(t)$:

$$c_{-n} = c_n, \text{Im}\{c_n\} = 0$$

- Real & odd $x(t)$:

$$c_{-n} = -c_n, \text{Re}\{c_n\} = 0$$

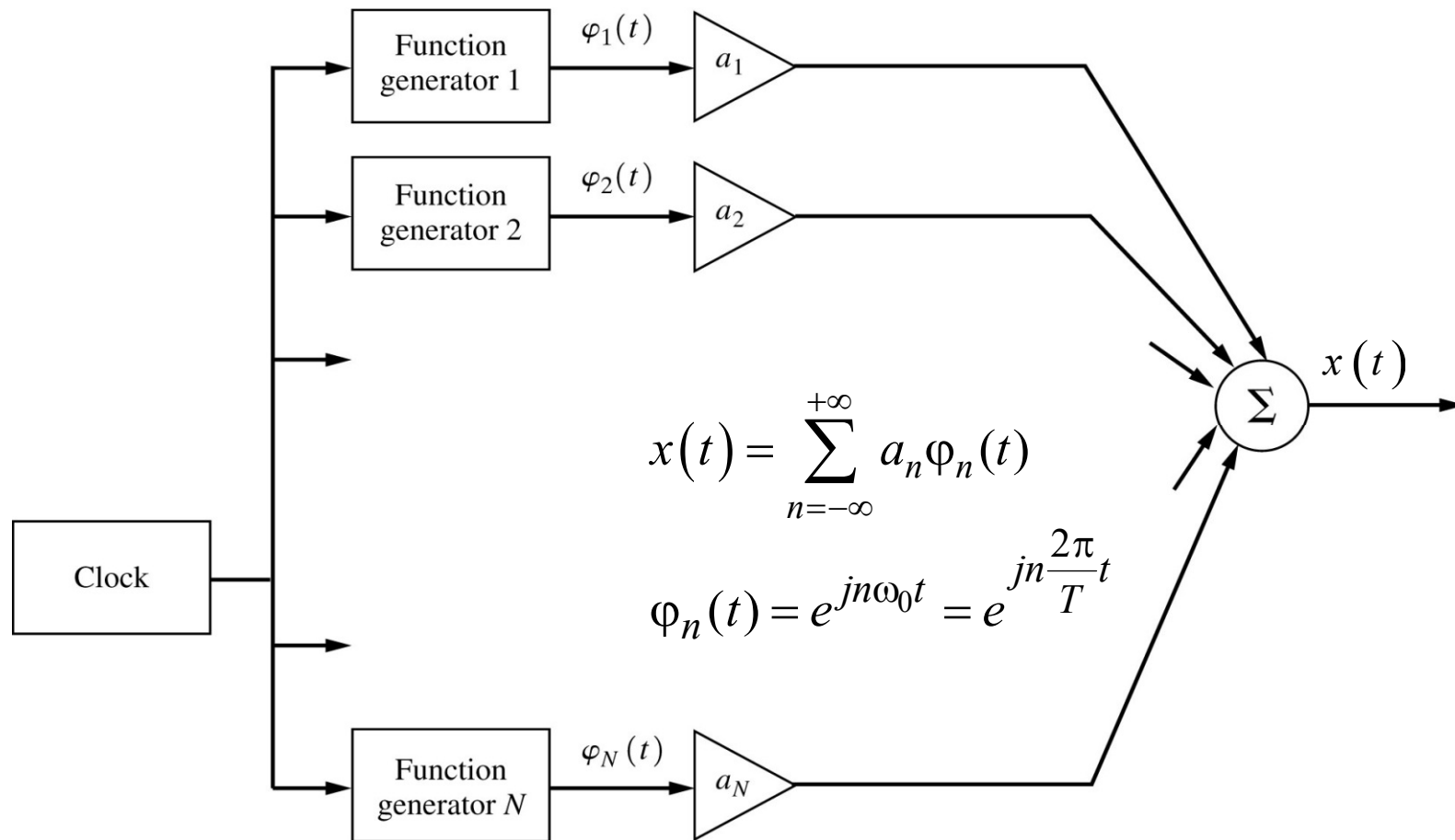
- Parseval's Theorem:

power 

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Q.: prove these properties

Signal Synthesis via FS



Couch, Digital and Analog Communication Systems, Seventh Edition.

Summary

- Review of Fourier series
- Periodic signals & complex exponents
- Series expansion of a periodic signal
- Trigonometric form of Fourier series
- Properties of Fourier series
- **Reading:** the Couch text, Sec. 2.1-2.5; Oppenheim & Willsky text, Sec. 3.0-3.5. Study carefully all the examples (including end-of-chapter study-aid examples), make sure you understand them and can solve them with the book closed.
- Do some end-of-chapter problems. Students' solution manual provides solutions for many of them.