Assignment 4, due: Apr. 23, 2pm (the final exam place/time) (hard copy, + scanned pdf via email)

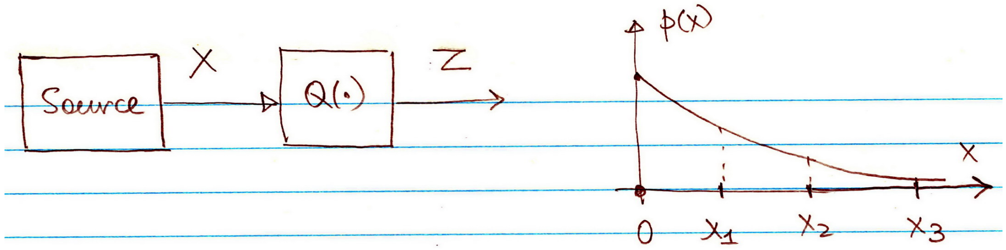
Note 1: Late submissions will not be accepted. Please also keep an extra copy of your submission for your own record.

Note 2: Submission format: can be hand-written or printed (handwriting must be readable), letter-sized paper, stapled, cover page must include course code/title, assignment number, your name and student number, date of submission, all pages numbered. Scanned pdf must be 100% identical to the hard copy. Please include all details/steps in your answers (e.g. derivations, analytical steps, etc.) not just final results. All symbols used must be clearly defined.

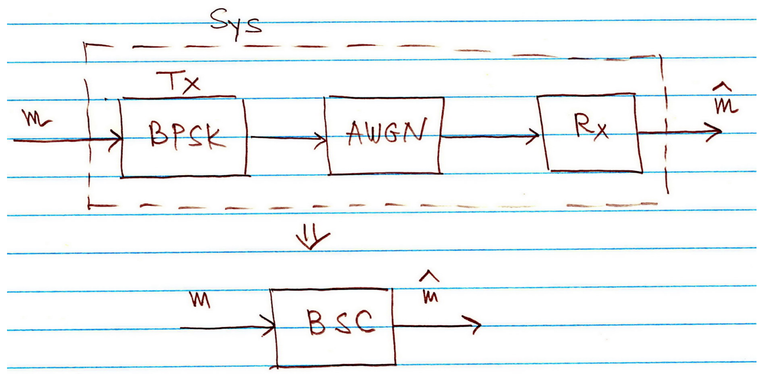
Note 3: Before doing the assignment, read relevant chapters/sections of the textbook (or other recommended books), study carefully all examples, attempt some end of chapter problems. Extra references are given below. No submission is needed for this part (this is very helpful for your studies though).

1. A digital image of 1000×2000 pixels and 24b color depth have to be stored on a hard disk.
   (a) What is the file size, in bits, to store this image in a raw (pixel-by-pixel, uncompressed) format?
   (b) Now assume that the best possible compression algorithm is applied to this file to compress it losslessly. What is the size of the compressed file? Assume that all pixels are iid random variables with the following probability distribution:
   $$p_m = 2^{-m}, m = 1, 2 \ldots 2^{24} - 1 \qquad (1)$$
   and find $p_{2^{24}}$ such that $\{p_m\}$ is a valid probability distribution. Compare this with uncompressed file size and explain the difference, if any.
   (c) Now assume that the whole image is made of 2 different (and random) colors only. How does the answer to the previous part change? Why?

2. A source is producing a continuous (analog) random variable $X \geq 0$ with the following pdf: $p(x) = \alpha e^{-\alpha x}$, $\alpha > 0$. Subsequently, it is quantized into 4 levels $x_0 = 0, x_1, x_2, x_3$ as shown below, where the quantizer $Q(\cdot)$ simply rounds off X to the nearest allowed level,
   $$z \triangleq Q(x) = x_m,\ m = \arg\min_i \{|x - x_i|\} \qquad (2)$$
   Find the the levels $\{x_1, x_2, x_3\}$ so that the entropy $H(Z)$ of quantizer output Z is maximized.



3. An engineer designs a communication system that transmits binary message m using BPSK modulation, as shown below; it operates in AWGN channel and uses an optimal receiver Rx.



   (a) Show that the whole system, from original message m to estimated message $\widehat{m}$ (this includes the Tx, the channel and the Rx), can be modeled as a binary symmetric channel (BSC). Find its cross-over probability, assuming that the SNR γ is known. Provide clear and rigorous arguments, not just verbal statements without justifications (you may use the results we discussed in class).

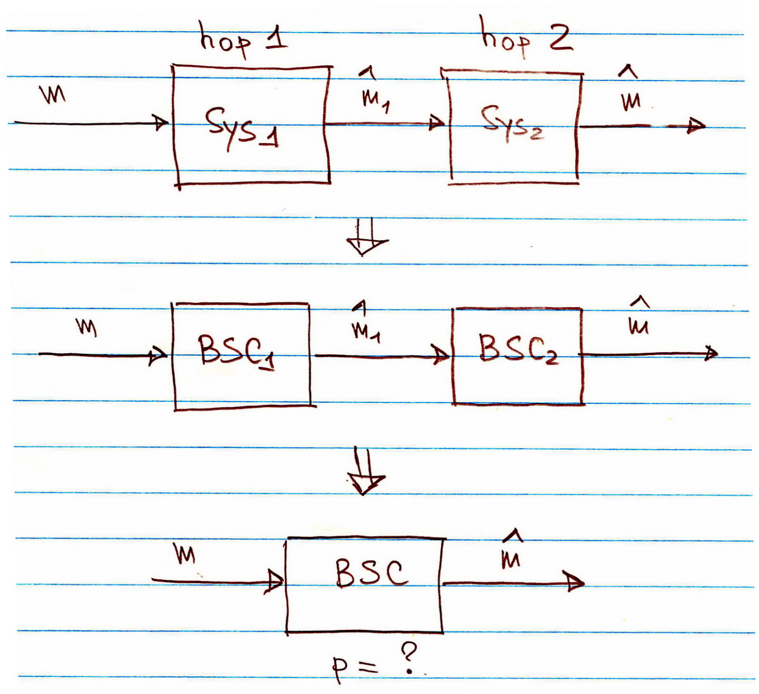
   (b) Using the previous part, find the system capacity C_{sys} (i.e. the maximum possible data rate subject to reliability constraint, which can be attained with the best possible channel coding) and plot it as a function $C_{sys}(\gamma)$ of the SNR γ. In the same figure, plot the AWGN capacity $C_0(\gamma)$, compare the two, and explain the difference, if any. Use reasonable scales for both axes so that all tendencies are clearly visible.

4. Now, we consider a 2-hop system as shown below, where each hop is the BPSK system of the previous question. This is known as relaying, whereby the transmitted message is estimated by the Rx of 1st hop system and then transmitted again by the Tx of 2nd-hop system (this is especially popular and useful for optical-fiber systems to eliminate noise accumulation and is known as regenerative repeater).

   (a) Argue that the 2-hop system can be modeled as a cascade connection of 2 BSCc as shown below. Find their cross-over probabilities p_1 and p_2 assuming that per-hop SNRs are γ_1 and γ_2.

(b) Show that this cascade connection is equivalent to a single BSC. Find its cross-over probability $p(\gamma_1, \gamma_2)$ as a function of γ_1, γ_2. Plot $p(\gamma_1, \gamma_2)$ as a function of γ_1 for a fixed $\gamma_2 = -6$ dB, 0 dB, 6 dB (in the same figure), and compare this plot to that of a single-hop system (i.e. when there is no 2nd hop), also plotted in the same figure. Explain the difference, if any, and also the impact of γ_2. Use reasonable scales for both axes so that all tendencies are clearly visible.

(c) Using the previous part, find, plot and compare the capacity of the whole 2-hop system with that of the 1-hop system (of Question 3). What is the impact of γ_2 on the capacity? Which system is better, one or two-hop? Explain why. When is the impact of 2nd hop is negligible? Explain why.



For all relevant questions, please include Matlab codes in your submission.

References

[1] S. Haykin, Digital Communication Systems, Wiley, 2014.

[2] R.E. Ziemer, W.H. Tranter, Principles of Communications, Wiley, 2009.

[3] S. Haykin, Communication Systems, 4th Ed, Wiley, 2001.

Note on plagiarism

Plagiarism (presenting somebody's else solution/report as your own, where "somebody" means any source, including Internet (Google, ChatGPT, etc.), "cut-and-paste" from a student to a student, other forms of "borrowing" the material) is absolutely unacceptable and will be penalized. Each student must submit his own solutions. If two (or more) identical or almost identical sets of solutions are found (including those from Internet), each student involved receives 0 (zero) for that particular assignment. If this happens twice, the students involved receive 0 (zero) for the entire assignment component of the course in the marking scheme and the case will be send to the Dean's office for further investigation and further penalties.