MIMO Channel Capacity

When the T_x signal correlation matrix $\mathbf{P} = \mathbf{E}\left\{\mathbf{xx}^+\right\} \neq \mathbf{I}$, the MIMO capacity is

$$\mathbf{C} = \log\left|1 + \frac{1}{\sigma_0^2}\mathbf{HPH}^+\right| \tag{1}$$

If CSI (channel state information) is available at the T_x, $\mathbf{P}$ can be chosen to maximize C, subject to the total T_x power constrain:

$$\sum_{i=1}^{m} P_{ii} = tr(\mathbf{P}) \leq P_T \tag{2}$$

where P_{ii} is the i-th T_x power.

Consider the $m \times n$ MIMO channel,

$$\mathbf{y} = \mathbf{Hx} + \xi \tag{3}$$

Using the SVD of $\mathbf{H}$,

$$\mathbf{H} = \mathbf{U\Sigma V}^+ \tag{4}$$

where $\mathbf{U}$,$\mathbf{V}$ are $n \times n$ and $m \times m$ unitary matrices, $\mathbf{U}^+\mathbf{U} = \mathbf{V}^+\mathbf{V} = \mathbf{I}$, and

$$\mathbf{\Sigma} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5}$$

and $\Sigma_1 = diag\left[\sigma_1, \quad \sigma_2, \quad .. \quad \sigma_k\right]$ are non-zero singular values of $\mathbf{H}$.

Using (4) and (3),

$$\mathbf{y} = \mathbf{U\Sigma V}^+\mathbf{x} + \xi \quad , \quad \tilde{\mathbf{y}} = \mathbf{\Sigma}\tilde{\mathbf{x}} + \tilde{\xi} \tag{6}$$

$$\widetilde{y}_i = \sigma_i \widetilde{x}_i + \widetilde{\xi}_i \quad , \quad i = 1, 2..k \tag{7}$$

where $\tilde{\mathbf{y}} = \mathbf{U}^+\mathbf{y}, \quad \tilde{\mathbf{x}} = \mathbf{V}^+\mathbf{x}, \quad \tilde{\xi} = \mathbf{U}^+\xi$. Note that multiplication by a unitary matrix does not change statistics of a random vector and, hence, does not affect the mutual information and the capacity.

Hence, the channel in (7) has the same capacity as the original channel (3). But the channel in (7) is as of k independent sub-channels, with per-sub-channel SNR:

$$\tilde{\gamma} = \lambda_i \frac{P_i}{\sigma_0^2} \tag{8}$$

and $\lambda_i = \sigma_i^2$ are eigenvalues of $\mathbf{HH}^+$, and $P_i = P_{ii}$. Its capacity is

$$\mathbf{C} = \sum_{i=1}^{k} \log_2\left(1 + \lambda_i \frac{p_i}{\sigma_0^2}\right) \tag{9}$$

Optimum P_i can be found using water-filling technique as follows:

$$P_i = \left[\mu - \frac{\sigma_0^2}{\lambda_i} \right]_+ , \quad i = 1, 2, ..k_1$$
(10)

$$\sum_{i=1}^{k_1} P_i = P_T$$
(11)

where $\left[\mathbf{x}\right]_+ = \mathbf{x}$ if $\mathbf{x}>0$ and 0 otherwise; k_1 is the number of active eigenmodes (i.e. with non-zero P_i), and constant μ is found from (10). Note that (10) and (11) also give (implicitly) k_1.

Water-filling technique can be formulated as iterative algorithm as follows [1]:

1) order eigenvalues, set iteration index p=0

2) find μ as follows

$$\mu = \frac{1}{k-p}\left(p_T + \sigma_0^2 \sum_{i=1}^{k-p} \frac{1}{\lambda_i} \right) \qquad (12)$$

3) set P_i using (10) with $k_1 = k - p$

4) if there is zero P_i, set $p = p+1$, eliminate λ_i and go to step 2

5) finish when all P_i (i=1, 2,..k-p) are non-zero.

This algorithm gives all non-zero P_i . All the other P_i are zeros (i.e., those eigenmodes are not used).

Proof of the water-filling technique :
using Lagrange multipliers with the following goal function,

$$F = \sum_i \log(1 + \frac{\lambda_i P_i}{\sigma_0^2}) - \alpha\left(\sum_i P_i - P_T \right) \qquad (13)$$

$$\frac{dF}{dP_i} = 0, \quad \frac{dF}{d\alpha} = 0 \qquad (14)$$

where α is a Lagrange multiplier. From (14), one obtains (10) and (11).

Finally, optimum $\mathbf{P}$ is found using (4)

$$\mathbf{P} = \mathbf{VDV}^+ \qquad (15)$$

where $D = diag\left[p_1, p_2, ..p_{k_1}, 0...0 \right].$
(16)

Effect of T_x CSI on the Capacity [2]

Compare the MIMO channel capacity in 2 cases:

1) no T_x CSI (uninformed T_x-UT)

2) full T_x CSI (informed T_x-IT)

In case 1, the capacity is given by (9) with $P_i = \frac{P_T}{m}$

$$\mathbf{C}_{UT} = \sum_{i=1}^{k} \log(1 + \frac{P_T}{m\sigma_0^2}\lambda_i) \quad (17)$$

In case 2, the capacity is given by (9) with P_i given by (10)

$$C_{IT} = \sum_{i=1}^{k_1} \log(1 + \frac{\lambda_i P_i}{\sigma_0^2}) \quad (18)$$

$$P_i = \left[\mu - \frac{\sigma_0^2}{\lambda_i} \right]_+ \quad (18a)$$

$$\sum_{i=1}^{k_1} P_i = P_T \quad (18b)$$

Consider the ratio

$$\beta = \frac{C_{IT}}{C_{UT}} \quad (19)$$

when $P_T / \sigma_0^2 \to \infty$ i.e. high SNR mode.

Assuming that P_T=const and $\sigma_0^2 \to \infty$, it is clear from (18a) and (18b) that $P_i=P_T/m$ (assuming k=m, i.e. full-rank channel), and

$$\frac{C_{IT}}{C_{UT}} \to 1 \quad as \quad \frac{P_T}{\sigma_0^2} \to \infty \quad (20)$$

Hence, optimum power allocation does not provide advantage in high SNR mode – parallel transmission (spatial multiplexing) with equal powers is optimum.

Consider the case of low SNR, $P_T / \sigma_0^2 \to 0$. Assume that $\sigma_0^2 \to \infty$, then from (18)-(18b) one finds that $P_{i_{\max}} = P_T$ and all the other $P_i = 0^*$, where $i_{\max}$ is the largest eigenmode index.

Hence,

$$C_{IT} = \log\left(1 + \frac{\lambda_{\max} P_T}{\sigma_0^2}\right) \approx \frac{\lambda_{\max} P_T}{\sigma_0^2} \log e \quad (21)$$

Similarly,

$$C_{UT} \approx \frac{P_T}{m\sigma_0^2} \sum_{i=1}^{m} \lambda_i \log e$$

Hence,

$$\frac{C_{IT}}{C_{UT}} \approx \frac{m\lambda_{\max}}{\sum_{i=1}^{m} \lambda_i} = \frac{m\lambda_{\max}(\mathbf{HH}^+)}{tr(\mathbf{HH}^+)} \quad (22)$$

Important conclusion: in low SNR case, the best strategy is to use the largest eigenmode only → this is beamforming!

In high SNR mode, the best strategy is to use spatial multiplexing (parallel transmission on all eigenmodes).

References:

1) A. Paulraj, R. Nabar, D. Gore, Introduction to Space-Time Wireless Communications, Cambridge University Press, 2003.

2) G. Larsson, P. Stoica, Space-Time Block Coding for Wireless Communications, Cambridge University Press, 2003.