# Robust Beamformer: Diagonal Loading

MVDR beamformer (and the others) are sensitive to an AOA mismatch, which s may degrade performance substantially in real-world conditions when AOA is estimated from received signals.

We need some ways to solve this problem. In other words, we need to design a robust beamformer, i.e. insensitive to small errors in parameters. This is a very important problem in many areas of communications, signal processing and control.

There are many solutions (each one has its own advantages and disadvantages); we consider two of them: diagonal loading (DL) and LCMV (LCMP).

A good measure of the array sensitivity to mismatch is the sensitivity function:

$$T = |\mathbf{w}|^2 \tag{9.1}$$

Our design constraint is

$$T = |\mathbf{w}|^2 \le T_0. \tag{9.2}$$

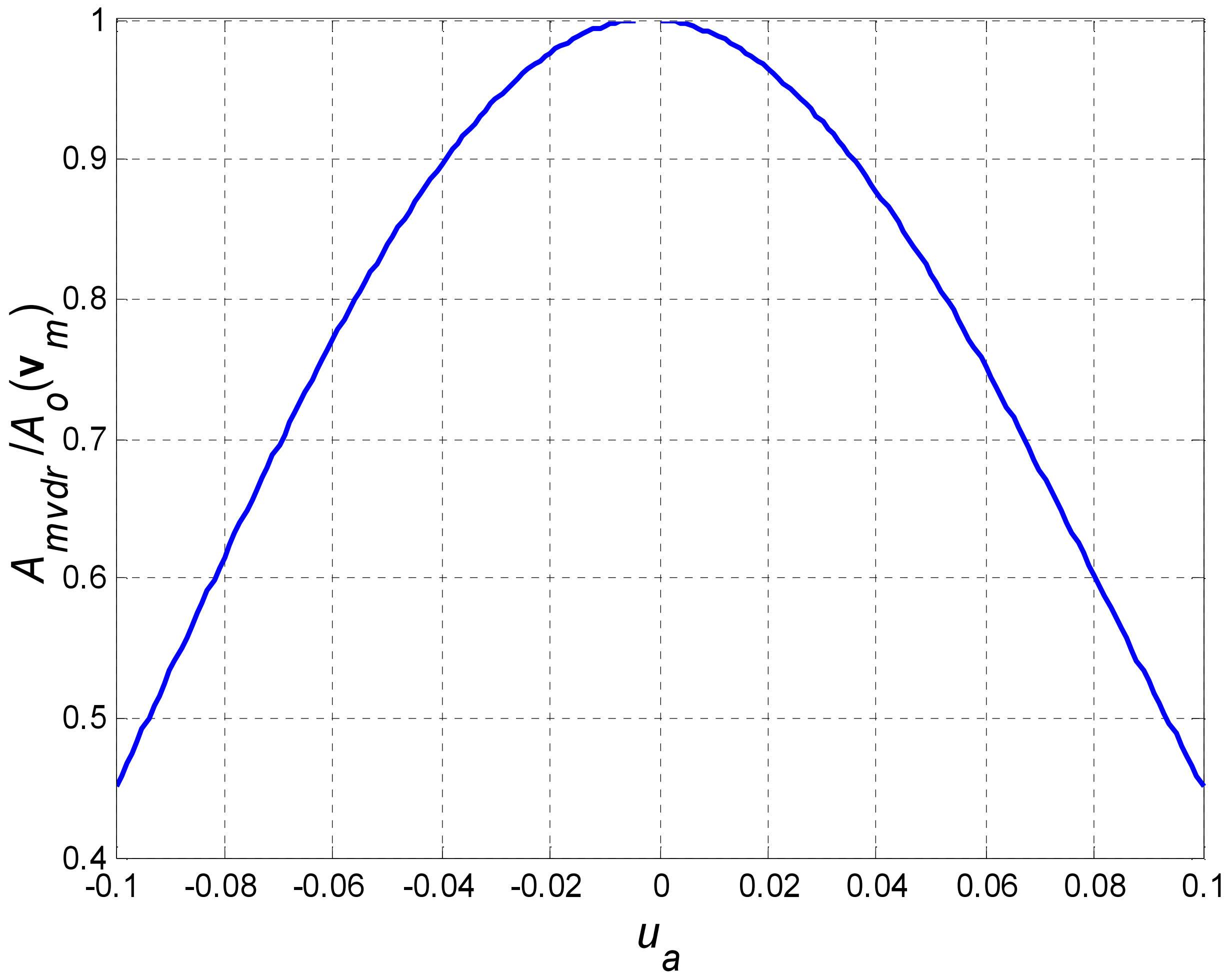
This limits the array sensitivity to all kinds of random variations, including the AOA mismatch.

Note: from the previous discussions,

$$T \geq \frac{1}{N} \tag{9.3}$$

So that one cannot do better than $T_0 = 1/N$.

# Example: AOA mismatch




H.L. Van Trees, Optimum Array Processing, Wiley, 2002

MVDR beamformer normalized gain vs. mismatch; $u = \cos\theta = \sin\bar{\theta}$;
$N = 10$; $d = \lambda/2$; $INR = 10dB$; $u_I = 0.3$, $u_a = \cos(\text{actual AOA})$

# Robust Beamformer via Optimization

Let us consider the following optimization problem (MPDR beamformer with limited sensitivity, i.e. a robust beamformer):

$$\begin{aligned} &\text{minimize } \mathbf{w}^{+}\mathbf{S}_x\mathbf{w} \\ &\text{subject to } \mathbf{w}^{+}\mathbf{v}_s = 1 \text{ and } \mathbf{w}^{+}\mathbf{w} = T_0 \end{aligned} \tag{9.5}$$

It can be solved using the Lagrange multiplier technique (as before). The Lagrangian is

$$F = \mathbf{w}^{+}\mathbf{S}_x\mathbf{w} + \lambda_1\left[\mathbf{w}^{+}\mathbf{w} - T_0\right] + \lambda_2\left[\mathbf{w}^{+}\mathbf{v}_s - 1\right] + \lambda_2^{*}\left[\mathbf{v}_s^{+}\mathbf{w} - 1\right] \tag{9.6}$$

Taking the derivative $\partial F / \partial \mathbf{w}$ and setting it to zero gives

$$\mathbf{w}_0^{+} = \frac{\mathbf{v}_s^{+}(\mathbf{S}_x + \sigma_L^2\mathbf{I})^{-1}}{\mathbf{v}_s^{+}(\mathbf{S}_x + \sigma_L^2\mathbf{I})^{-1}\mathbf{v}_s} \tag{9.7}$$

where $\sigma_L^2 = \lambda_1$ is a design parameter.

Note that the effect of the quadratic constraint (QC) $\mathbf{w}^+\mathbf{w} = T_0$ is to add the diagonal matrix $\sigma_L^2\mathbf{I}$ to $\mathbf{S}_x$. In fact, it means that we design a beamformer for a higher noise level than is actually present, as seen from

$$\mathbf{S}_x' = \mathbf{S}_x + \sigma_L^2\mathbf{I} = \mathbf{S}_{x_s} + \mathbf{S}_I + \left(\sigma_0^2 + \sigma_L^2\right)\mathbf{I} \tag{9.8}$$

*Lagrange multiplier technique gives a robust solution with minimal effort!*

Q.: prove that (9.7) delivers minimum, not maximum.

Q.: explain why DL works (i.e. why robust)!

Note that as $\sigma_L^2 \to \infty$, the MPDR-DL beamformer approaches the conventional beamformer.

Q.: explain why!

Important design parameter – the loading-to-noise ratio (LNR):

$$LNR = \sigma_L^2 / \sigma_0^2 \tag{9.9}$$

Performance will significantly depend on it.

By varying LNR, we may maximize the array gain for given SNR and INR.
Approximate rule of thumb:

$$SNR + 10dB \le LNR \le INR \tag{9.10}$$

If LNR>INR, the interference is not canceled adequately
If LNR<SNR+10dB, the effect of diagonal loading is small and variation effects are not cancelled.

Note: the approach is feasible when

$$INR \ge SNR + (10...15)\,dB \tag{9.11}$$

DL provides good measure against random variations in array elements.

We can select appropriate LNR only if we have reasonably good information about expected SNR and INR.

Q: what to do if we don't have it?

Overall, DL plays an important role in the design of robust beamformers.

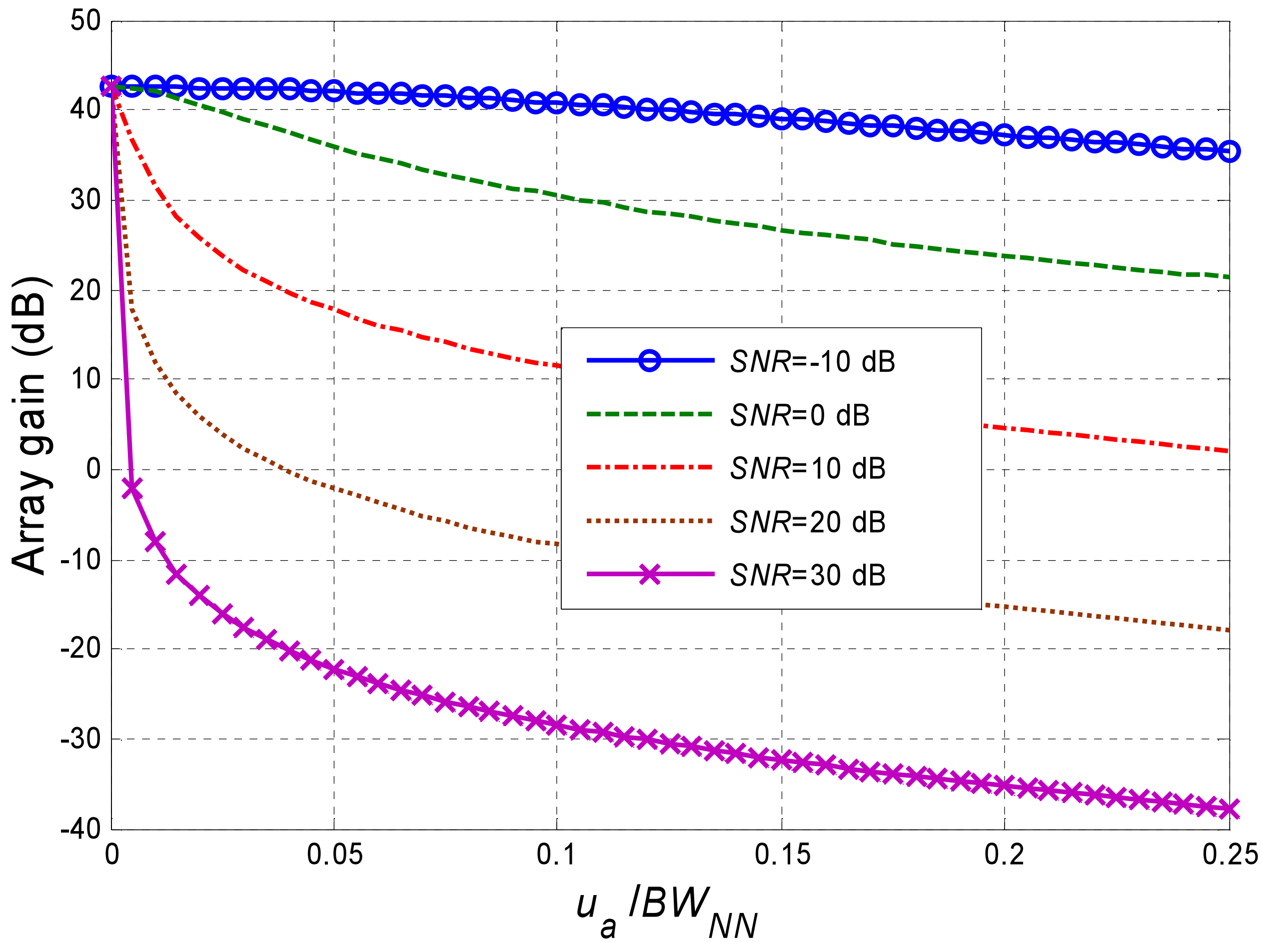
See Van Trees, section 6.6 for extensive discussion.

# DL: The array gain vs. AOA mismatch

*LNR*=0 dB

$$SNR=\sigma_s^2/\sigma_0^2,\ N=10;$$
$$d=\lambda/2;\ INR=30dB;$$
$$u_I=\pm 0.3,$$
$$u=\cos\theta=\sin\bar{\theta}$$
$$u_a=\cos(\text{actual AOA})$$
$$BW_{NN}=\frac{2\lambda}{Nd}$$

- null-to-null beamwidth in the $u$-space.

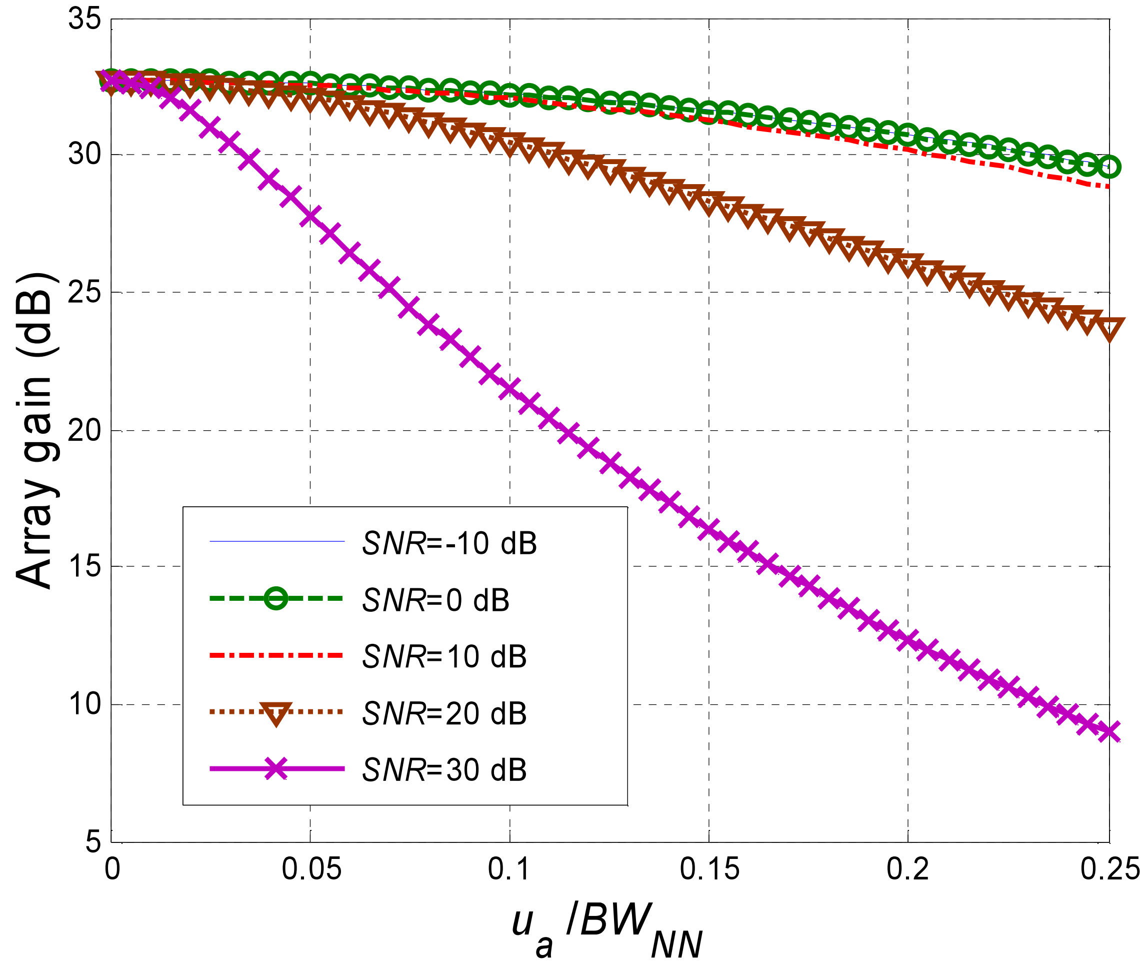



H.L. Van Trees, Optimum Array Processing, Wiley, 2002

Q: explain why it drops so fast at high SNR (non-trivial).

# DL: The array gain vs. AOA mismatch

the same parameters as above, except for the LNR=30 dB

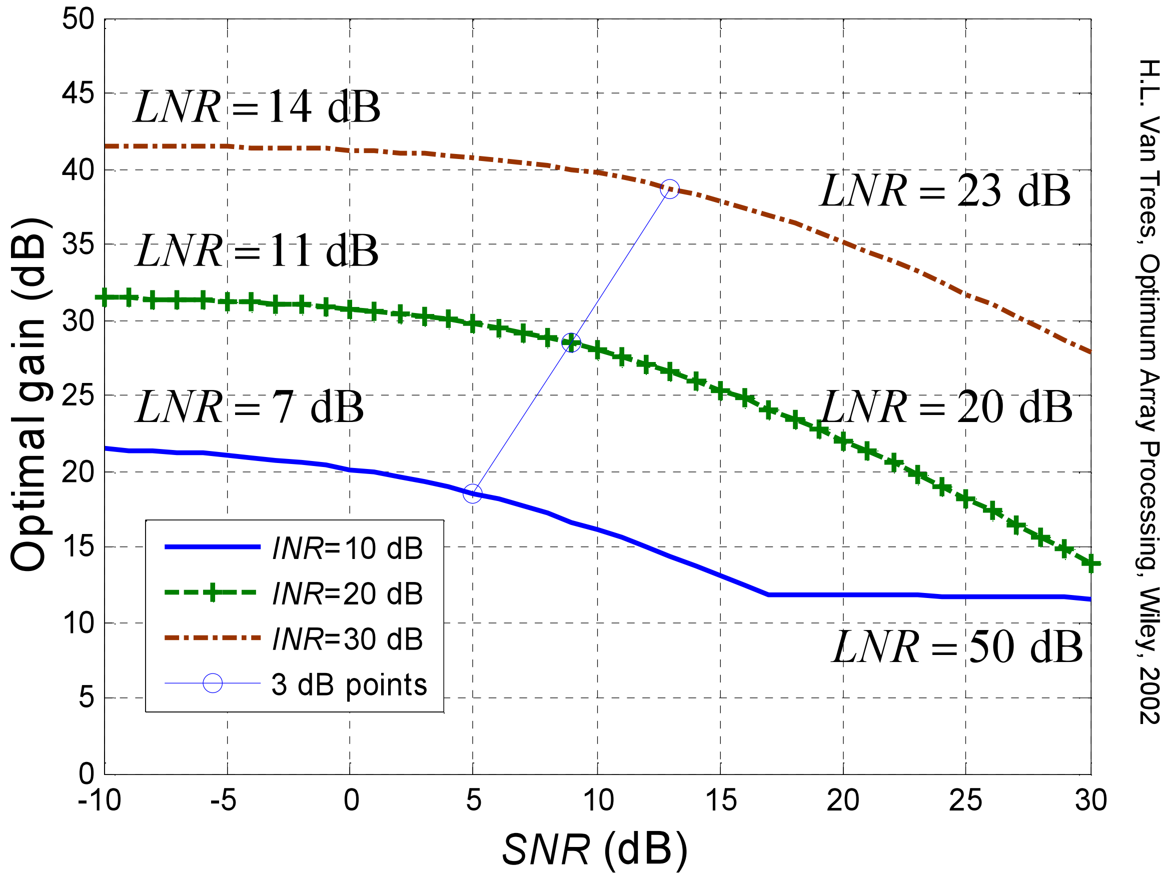



H.L. Van Trees, Optimum Array Processing, Wiley, 2002

Q: compare to the previous one and explain the difference.

# Optimal Loading:The optimum gain vs. SNR

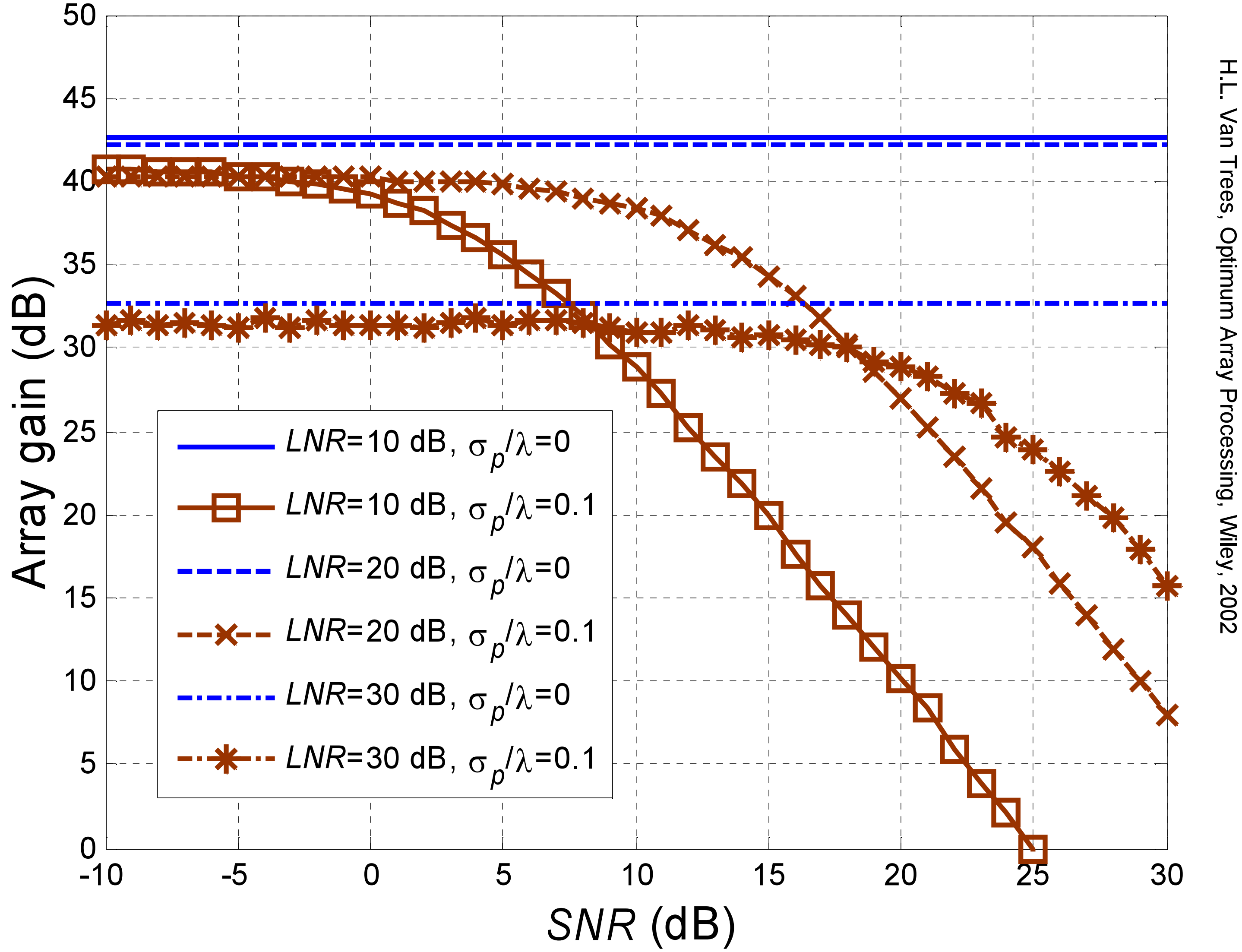
In this example, the optimum LNR is found by computing the gain as a function of LNR and then maximizing it (see Van Trees, sec. 6.6.4 for details). The optimum LNR is a function of SNR. The array parameters are the same as above.




$|u_a| \le 0.1$, random, uniformly distributed.     Q: explain the curves!

# DL: The array gain vs. SNR for an array with perturbations

The array parameters are the same as above, INR=30 dB



# Summary

- Robust beamformer.
- Diagonal loading provides robustness against various mismatches and perturbations.
- DL: design for more noise than actually present.

## References

1. H.L. Van Trees, Optimum Array Processing, Wiley, New York, 2002, Chapter 6.6.

## Homework

Fill in the details in the derivations above. Answer the questions. Do the examples yourself.