Maximum SNR Beamforming

Consider the same model as before: $\mathbf{x} = \mathbf{x}_s + \boldsymbol{\xi}$.

Note: we don't assume $\mathbf{x}_s$ is a plane wave -> generic case.

The output signal power is

$$\sigma_{s,out}^2 = \left\langle x_{s,out} x_{s,out}^* \right\rangle = \mathbf{w}^+ \mathbf{S}_s \mathbf{w} \quad (8.1)$$

The output noise power

$$\sigma_{\xi,out}^2 = \left\langle \xi_{out} \xi_{out}^* \right\rangle = \mathbf{w}^+ \mathbf{S}_\xi \mathbf{w} \quad (8.2)$$

The output SNR

$$\beta = SNR_{out} = \frac{\sigma_{s,out}^2}{\sigma_{\xi,out}^2} = \frac{\mathbf{w}^+ \mathbf{S}_s \mathbf{w}}{\mathbf{w}^+ \mathbf{S}_\xi \mathbf{w}} \quad (8.3)$$

The optimum value of $\mathbf{w}$ is obtained by taking the derivative of β w.r.t. $\mathbf{w}^+$ and setting it to zero,

$$\nabla_{\mathbf{w}^+} \beta = 0$$

and

$$\nabla_{\mathbf{w}^+} \beta = \frac{\partial \beta}{\partial \mathbf{w}^+} = \frac{\mathbf{S}_s \mathbf{w} \mathbf{w}^+ \mathbf{S}_\xi \mathbf{w} - \mathbf{w}^+ \mathbf{S}_s \mathbf{w} \mathbf{S}_\xi \mathbf{w}}{\left(\mathbf{w}^+ \mathbf{S}_\xi \mathbf{w}\right)^2} = 0 \quad (8.4)$$

It follows that this is an eigenvalue problem,

$$\mathbf{S}_\xi^{-1} \mathbf{S}_s \mathbf{w} = \beta \mathbf{w}$$

which can be presented as follows,

$$\mathbf{A}\mathbf{w} = \beta\mathbf{w}, \ \mathbf{A} = \mathbf{S}_{\xi}^{-1}\mathbf{S}_s \qquad (8.5)$$

Since we want to maximize β, the solution is the eigenvector with the largest eigenvalue.

This can be done numerically for any $\mathbf{S}_{\xi}^{-1}\mathbf{S}_s$, i.e. for any signal and noise + interference.

The largest eigenvalue becomes SNR_{out}.

Plane-wave signal:

$$\mathbf{x}_s = x_s \mathbf{v}_s, \ \ \mathbf{S}_s = \sigma_s^2 \mathbf{v}_s \mathbf{v}_s^+ \qquad (8.6)$$

$$\sigma_s^2 \mathbf{S}_{\xi}^{-1} \mathbf{v}_s \mathbf{v}_s^+ \mathbf{w} = \gamma \mathbf{w} \qquad (8.7)$$

Since $\mathbf{v}_s^+\mathbf{w}$ is a scalar, (8.7) reduces to $\mathbf{w} = a\mathbf{S}_{\xi}^{-1}\mathbf{v}_s$.

Note that β does not change if $\mathbf{w}$ is multiplied by a scalar (which may be frequency-dependent). Hence, we may assume that a=1 and the maximum SNR beamformer is

$$\mathbf{w}_0^+ = \mathbf{v}_s^+ \mathbf{S}_{\xi}^{-1} \qquad (8.8)$$

Comparison of MVDR, MMSE and Max. SNR Beamformers

Max SNR	**MMSE**	**MVDR**
$\mathbf{w}_0^+ = \mathbf{v}_s^+ \mathbf{S}_\xi^{-1}$	$\mathbf{w}_0^+ = \dfrac{\sigma_s^2 \gamma}{\sigma_s^2 + \gamma} \mathbf{v}_s^+ \mathbf{S}_\xi^{-1}$	$\mathbf{w}_0^+ = \dfrac{\mathbf{v}_s^+ \mathbf{S}_\xi^{-1}}{\mathbf{v}_s^+ \mathbf{S}_\xi^{-1} \mathbf{v}_s}$

This is essentially the same beamformer, up to a scalar factor. Hence , MVDR and MMSE can use the eigen-decomposition techniques as well.

Note that all of them provide the same SNR_{out}:

$$SNR_{out} = \sigma_s^2 \mathbf{v}_s^+ \mathbf{S}_\xi^{-1} \mathbf{v}_s = \frac{\sigma_s^2}{\gamma} = SNR_{in} \cdot G_0 \qquad (8.9)$$

where $G_0 = \mathbf{v}_s^+ \mathbf{S}_{n\xi}^{-1} \mathbf{v}_s$ is the optimum array gain.

Reminder: $\gamma = \left[\mathbf{v}_s{}^+ \mathbf{S}_\xi^{-1} \mathbf{v}_s \right]^{-1}$.

Minimum Power Distortionless Response (MPDR) Beamformer

Consider the same model as before: $\mathbf{x} = \mathbf{x}_s + \xi$.

The basic idea is to minimize the total power at the output,

$$P_{out} = \left\langle \left| \mathbf{w}^+ \mathbf{x} \right|^2 \right\rangle = \mathbf{w}^+ \mathbf{S}_x \mathbf{w} \qquad (8.10)$$

subject to distortionless constraint

$$\mathbf{w}^+ \mathbf{v}_{ex} = 1$$

where $\mathbf{v}_{ex}$ is the array manifold vector for expected signal direction, $\mathbf{v}_{ex} = \mathbf{v}(\mathbf{k}_{ex})$. Consider the case when $\mathbf{k}_{ex}$ may not be equal to the actual signal direction. This is signal mis-match problem.

Using the same argument as for the MVDR beamformer, we obtain,

$$\mathbf{w}_0^+ = \frac{\mathbf{v}_{ex}^+ \mathbf{S}_x^{-1}}{\mathbf{v}_{ex}^+ \mathbf{S}_x^{-1} \mathbf{v}_{ex}} \qquad (8.11)$$

Note that $\mathbf{S}_x^{-1}$ is used, rather than $\mathbf{S}_\xi^{-1}$ (as for the MVDR). However, when $\mathbf{v}_{ex} = \mathbf{v}_s$, the MPDR and MVDR weights are equal,

$$\frac{\mathbf{v}_s^+ \mathbf{S}_x^{-1}}{\mathbf{v}_s^+ \mathbf{S}_x^{-1} \mathbf{v}_s} = \frac{\mathbf{v}_s^+ \mathbf{S}_\xi^{-1}}{\mathbf{v}_s^+ \mathbf{S}_\xi^{-1} \mathbf{v}_s}$$

Q: prove it!

This approach allows us to estimate the effect of mismatch: $\mathbf{v}_{ex} \neq \mathbf{v}_s$.

Since it is easier to measure $\mathbf{S}_x$, this version is used more frequently in practice. In the literature, MVDR is often referred to both versions.

General remark: we see that for a wide range of criteria, the optimum beamformer is MVDR followed by a scalar filter,

$$\mathbf{w}_o^+ = a\mathbf{v}_s^+\mathbf{S}_\xi^{-1}$$

Performance of Optimal Beamformers

We consider the case of plane-wave interfering signals +white noise, and assume that $N_I < N-1$

where N_I is the number of interferers. We will see that all the processing is done in N_I+1 dimensional "signal+interference" subspace rather than original N dimensional space, this simplifies the problem significantly when $N_I+1 \ll N$

The basic result is that the beamformer places nulls in the directions of interfering signals.

Single plane-wave interferer

Since all the optimal beamformers are a scaled version of MVDR beamformer (for single plane-wave signal), we further consider MVDR.

The total noise correlation matrix (noise + interferer):

$$\mathbf{S}_\xi = \sigma_0^2\mathbf{I} + \sigma_1^2\mathbf{v}_1\mathbf{v}_1^+ \qquad (8.12)$$

Using the M.I.L.,

$$\mathbf{S}_\xi^{-1} = \frac{1}{\sigma_0^2}\left[\mathbf{I} - \frac{\sigma_1^2}{\sigma_0^2 + N\sigma_1^2}\mathbf{v}_1\mathbf{v}_1^+\right] \qquad (8.13)$$

The optimal weights

$$\mathbf{w}_o^+ = \frac{\gamma \mathbf{v}_s^+}{\sigma_0^2}\left[\mathbf{I} - \frac{\sigma_1^2}{\sigma_0^2 + N\sigma_1^2}\mathbf{v}_1\mathbf{v}_1^+\right] \qquad (8.14)$$

Define the spatial correlation coefficient

$$r_{s1} = \frac{\mathbf{v}_s^+ \mathbf{v}_1}{N} = F_c\left(\mathbf{k}_1 \middle| \mathbf{k}_s\right) \qquad (8.15)$$

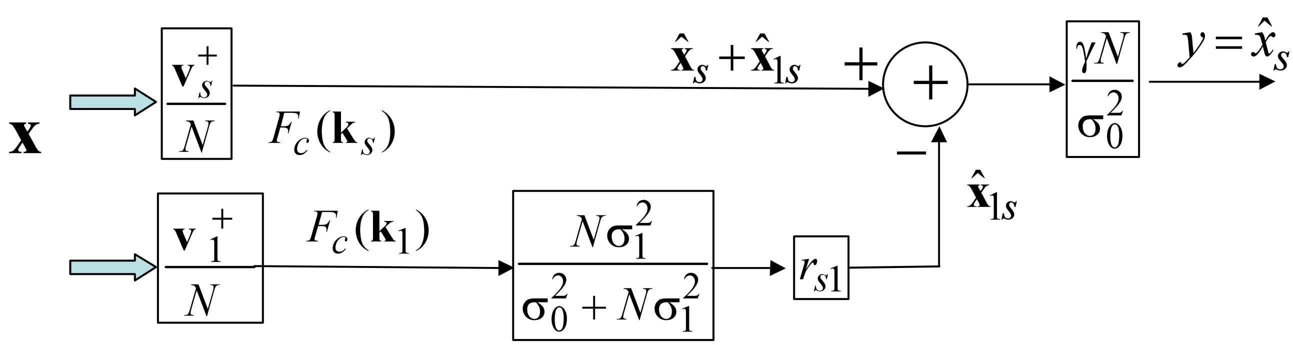
In fact, r_{s1} is the conventional beam pattern steered to $\mathbf{k}_s$ and evaluated at $\mathbf{k}_1$.

Then the optimum weights can be presented as

$$\mathbf{w}_0^+ = \frac{\gamma N}{\sigma_0^2}\left[\frac{\mathbf{v}_s^+}{N} - r_{s1}\frac{N\sigma_1^2}{\sigma_0^2 + N\sigma_1^2}\frac{\mathbf{v}_1^+}{N}\right] \qquad (8.16)$$

$$\gamma = \left[\frac{N}{\sigma_0^2}\left(1 - \frac{N\sigma_1^2}{\sigma_0^2 + N\sigma_1^2}\left|r_{s1}\right|^2\right)\right]^{-1} \qquad (8.17)$$

The resulting beamformer is



Note that the upper part forms a beam at $\mathbf{k}_s$, the lower part forms a beam at $\mathbf{k}_1$, and finally, the interferer contribution is subtracted out. This is a sidelobe cancellation.

The optimum array gain is

$$G_0 = \mathbf{v}_s^+ \mathbf{S}_{n\xi}^{-1} \mathbf{v}_s = N(1+\alpha)\left[\frac{1+N\alpha(1-|r_{s1}|^2)}{1+N\alpha}\right] \quad (8.18)$$

where $\alpha = \sigma_1^2 / \sigma_0^2$ is the interference-to-noise ratio (INR).

Q.: prove (8.18).

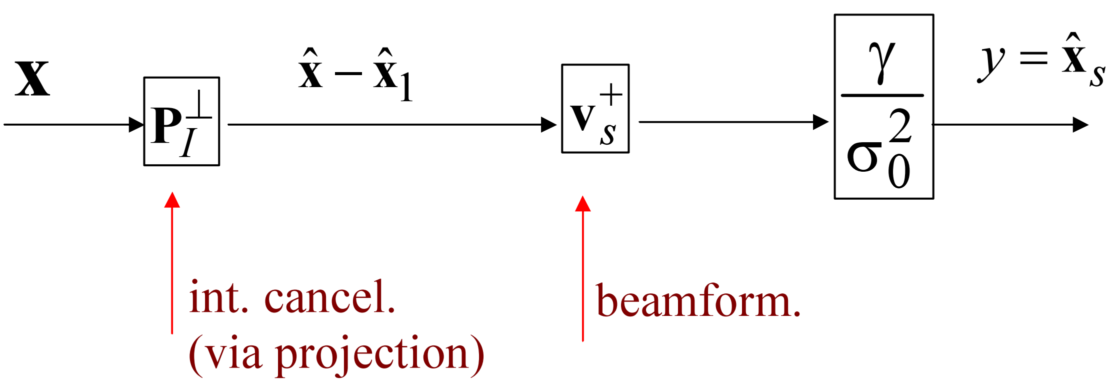
Note the G_0 depends on both α and $|r_{s1}|$ (and of course on N).

Consider the case of large INR, $N\alpha \gg 1$, then

$$\mathbf{w}_0^+ = \frac{\gamma}{\sigma_0^2} \mathbf{v}_s^+ \mathbf{P}_I^\perp, \quad \mathbf{P}_I^\perp = \mathbf{I} - \mathbf{v}_1 \left[\mathbf{v}_1^+ \mathbf{v}_1\right]^{-1} \mathbf{v}_1^+ \quad (8.19)$$

where $\mathbf{P}_I^\perp$ - projection matrix onto the subspace orthogonal to the interferer subspace.

The beamformer is



Optimum gain for $|r_{s1}|<1$ is $G_0 \approx N\alpha(1-|r_{s1}|^2)$

Q.: Explain why the gain is given by this expression when $|r_{s1}|=0$.

Note that when $|r_{s1}|=1$, $G_0 = \dfrac{N(1+\alpha)}{1+N\alpha} \approx 1$ for $\alpha \gg 1$.

In this case, the beamformer is not able to discriminate between the required signal and interferer.

Q: Why? What to do in this case?

Q: Consider (8.18) when $N\alpha \ll 1$ (low INR regime).

The output SNR is

$$SNR_{out} = SNR_{in}G_0 = \frac{\sigma_s^2}{\gamma} = N\frac{\sigma_s^2}{\sigma_0^2}\left(1-\frac{N\sigma_1^2}{\sigma_0^2+N\sigma_1^2}|r_{s1}|^2\right)$$

$$\approx N\frac{\sigma_s^2}{\sigma_0^2}(1-|r_{s1}|^2) \text{ for } \alpha \gg 1 \tag{8.20}$$

When $|r_{s1}|=1$,

$$SNR_{out} = \frac{N\sigma_s^2}{\sigma_0^2+N\sigma_1^2} \approx \frac{\sigma_s^2}{\sigma_1^2} \tag{8.21}$$

Hence in this case the beamformer is not able to cancel the interference.

When $|r_{s1}| < 1$ and $N\sigma_1^2 \gg \sigma_0^2$,

$$SNR_{out} = N \cdot \frac{\sigma_s^2}{\sigma_0^2}(1 - |r_{s1}|^2) \qquad (8.22)$$

Note that if $|r_{s1}| = 0$, the interference is cancelled completely,

$$SNR_{out} = N \cdot \frac{\sigma_s^2}{\sigma_0^2} \qquad (8.23)$$

as if there would be no interference at all. The beamformer places a perfect null at $\mathbf{k}_1$.

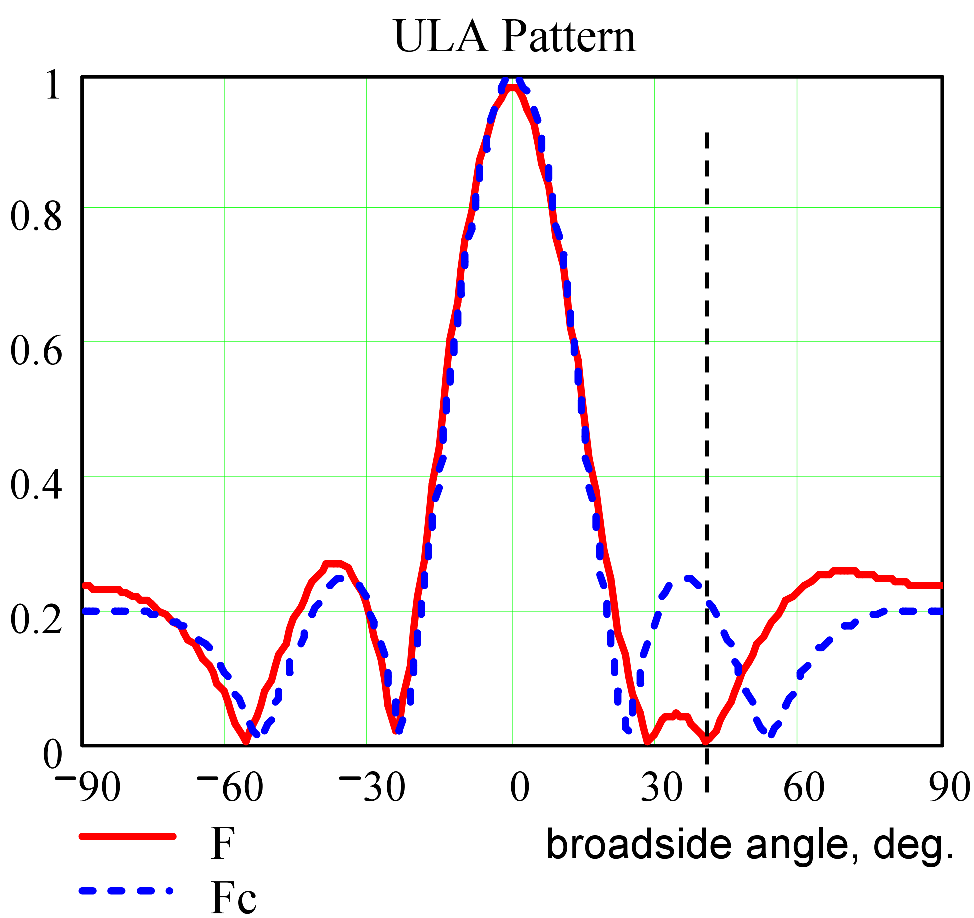
Q.: Explain why (8.23) corresponds to the case of no interference at all.

If $|r_{s1}| \neq 0$, the term $|r_{s1}|$ describes the residual interference at the output, going through the required signal channel.

Performance Examples: MMSE Beamformer

$$N = 5;\ d = \lambda/2;\ \theta_s = 0^o;\ \theta_I = 40^o;$$

$$SNR = \frac{\sigma_s^2}{\sigma_0^2} = 10;\ INR = \frac{\sigma_1^2}{\sigma_0^2} = 10;$$



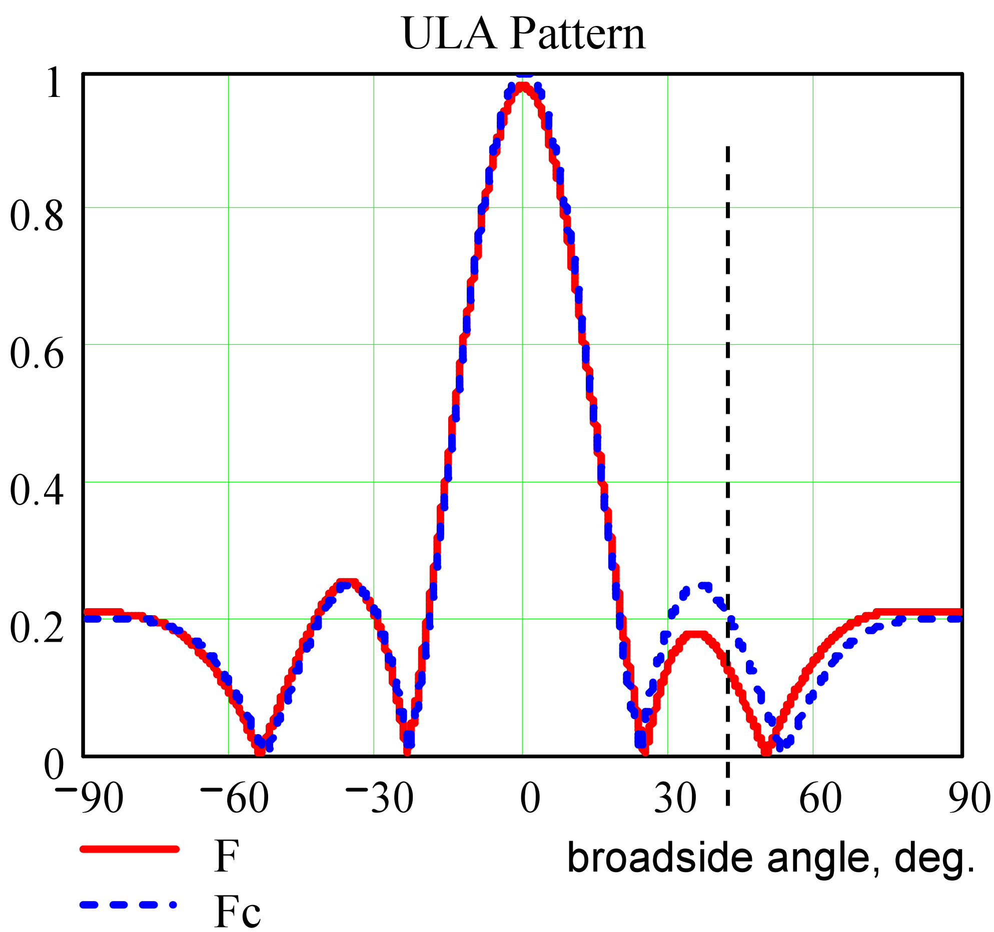
$$SNIR_{in} = 0.9;\ SNIR_{out} = 47.6;$$

$$G_0 = 52.3$$ <- explain this!

Performance Examples: MMSE Beamformer

$$N = 5;\ d = \lambda / 2;\ \theta_s = 0^o;\ \theta_I = 40^o;$$

$$SNR = \frac{\sigma_s^2}{\sigma_0^2} = 10;\ INR = \frac{\sigma_1^2}{\sigma_0^2} = 0.1;$$

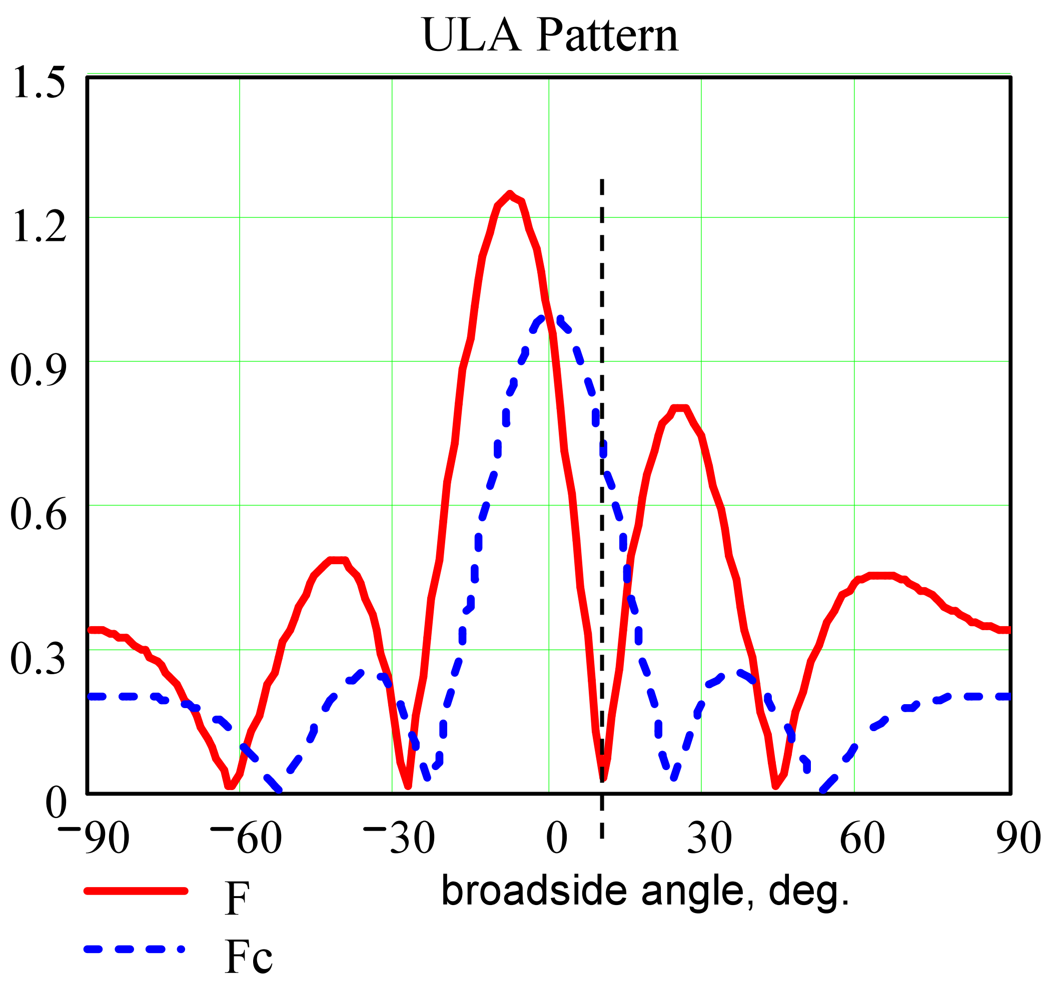


$$SNIR_{in} = 9;\ SNIR_{out} = 49;$$

$G_0 = 5.4$ <- explain this!

Performance Examples: MMSE Beamformer

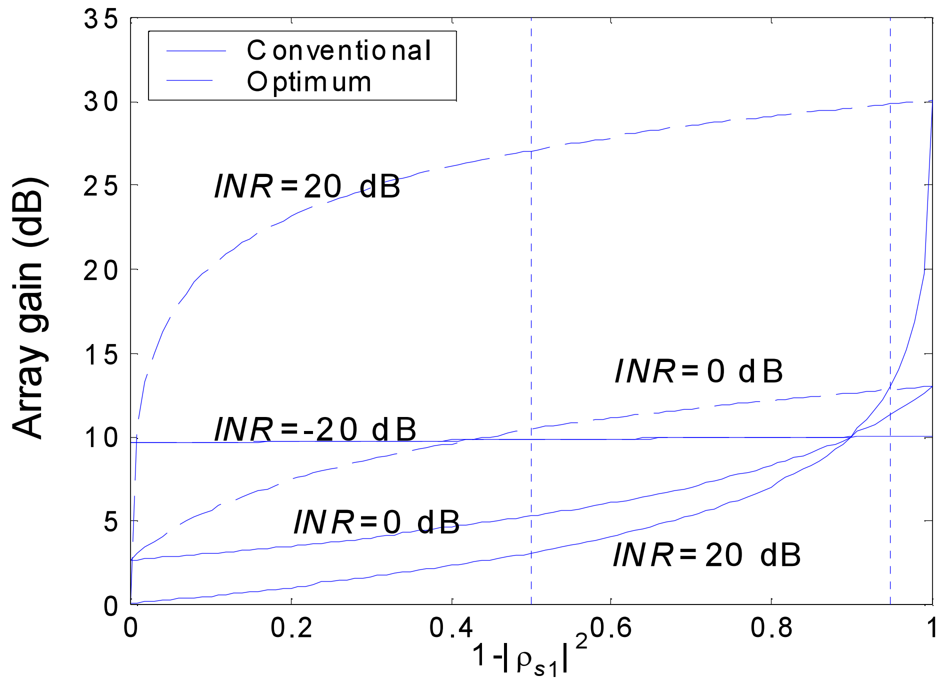
$$N=5;\ d=\lambda/2;\ \theta_s=0^o;\ \theta_I=10^o;$$
$$SNR=10;\ INR=10;$$

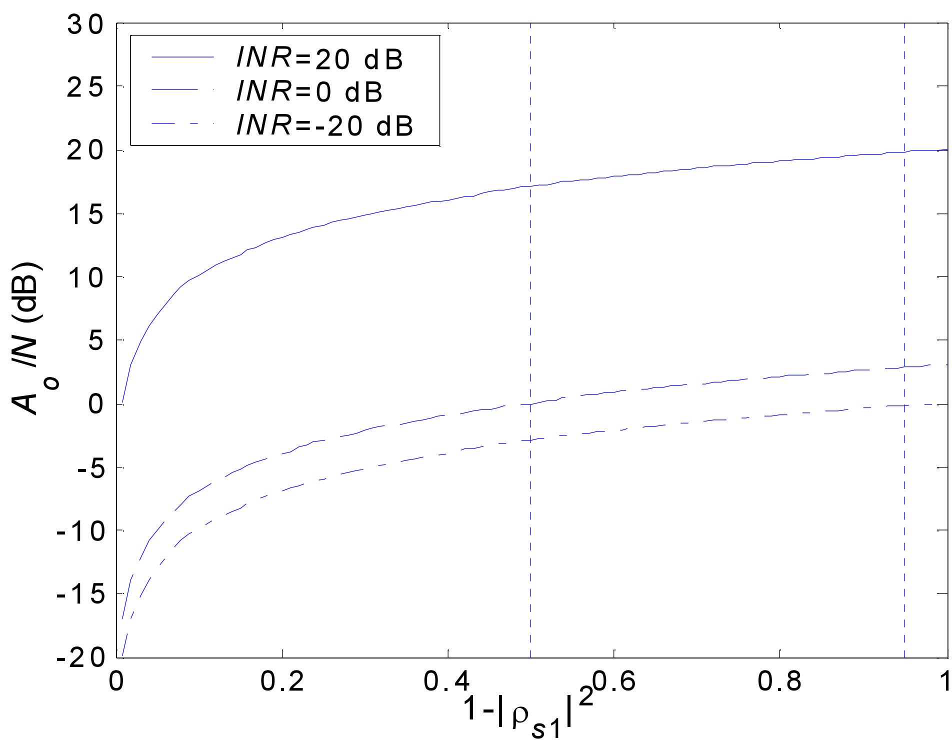


$$SNIR_{in}=0.9;\ SNIR_{out}=24.1;$$
$$G_0=26.5$$

Performance: Examples

10 element array



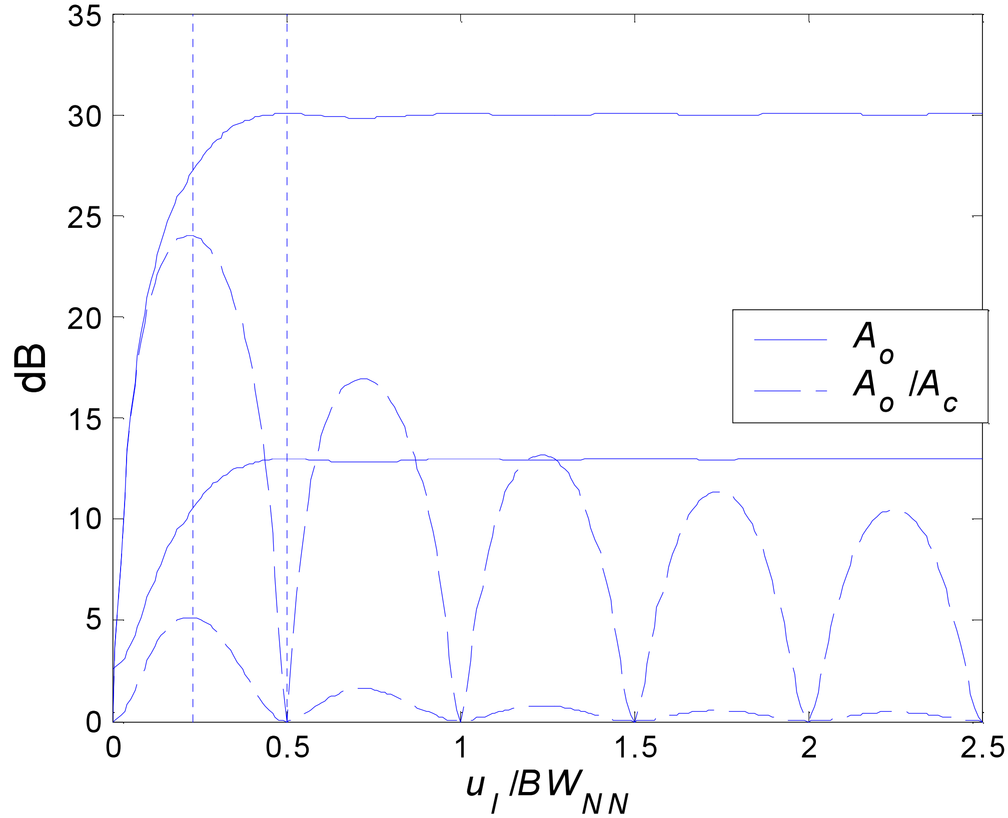


Explain the graphs!

H.L. Van Trees, Optimum Array Processing, Wiley

array gain vs. correlation

Performance: Examples



H.L. Van Trees, Optimum Array Processing, Wiley

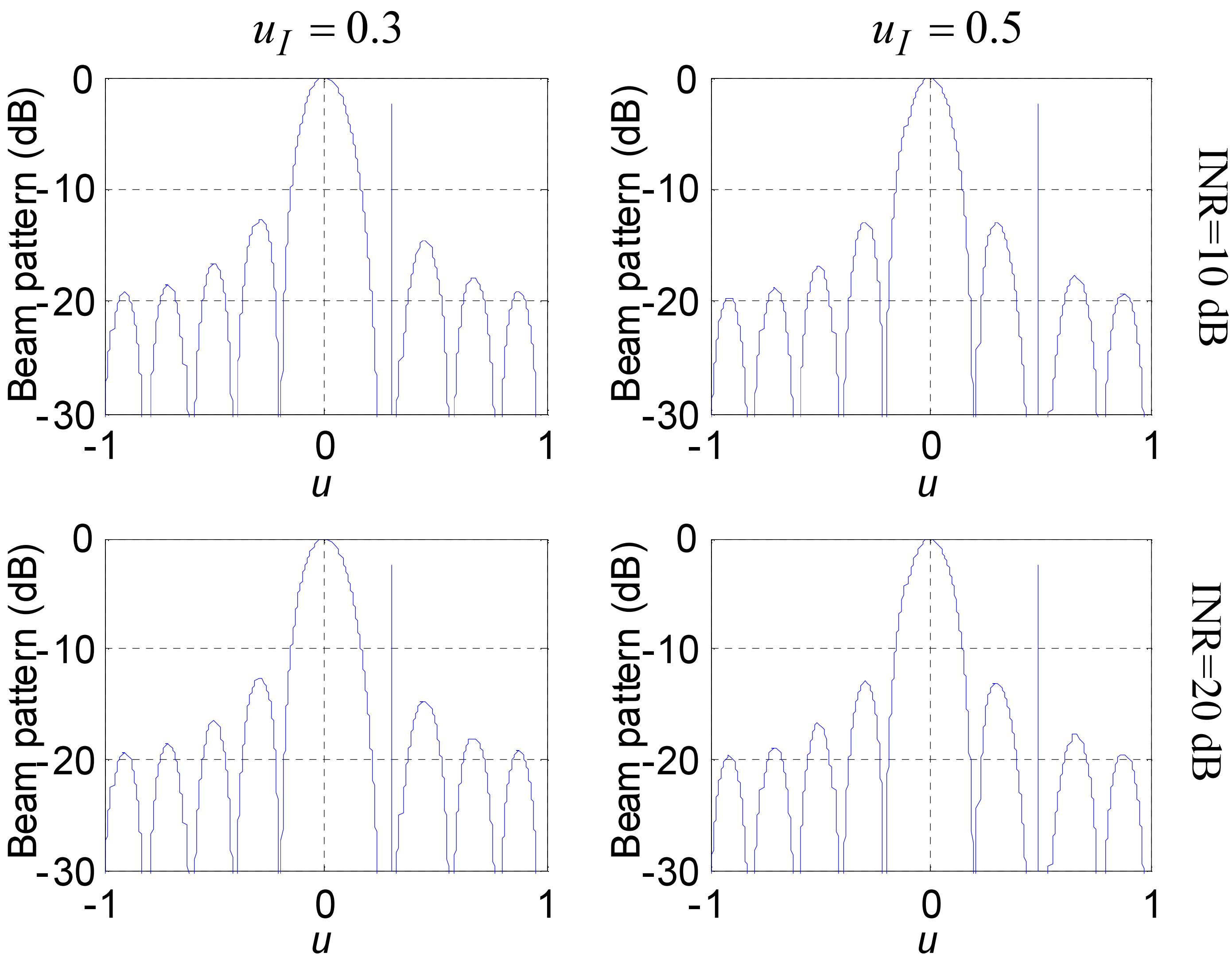
Optimum and conventional gains vs. interferer AOA

$INR = 0dB$ and $20dB$; $u = \cos\theta = \sin\bar{\theta}$

$BW_{NN} = \dfrac{2\lambda}{Nd}$ is the null-to-null beamwidth in $u = \sin\bar{\theta}$

A_0=N*INR when the interferer is out of the main beam

Performance: Examples, sidelobe interferer



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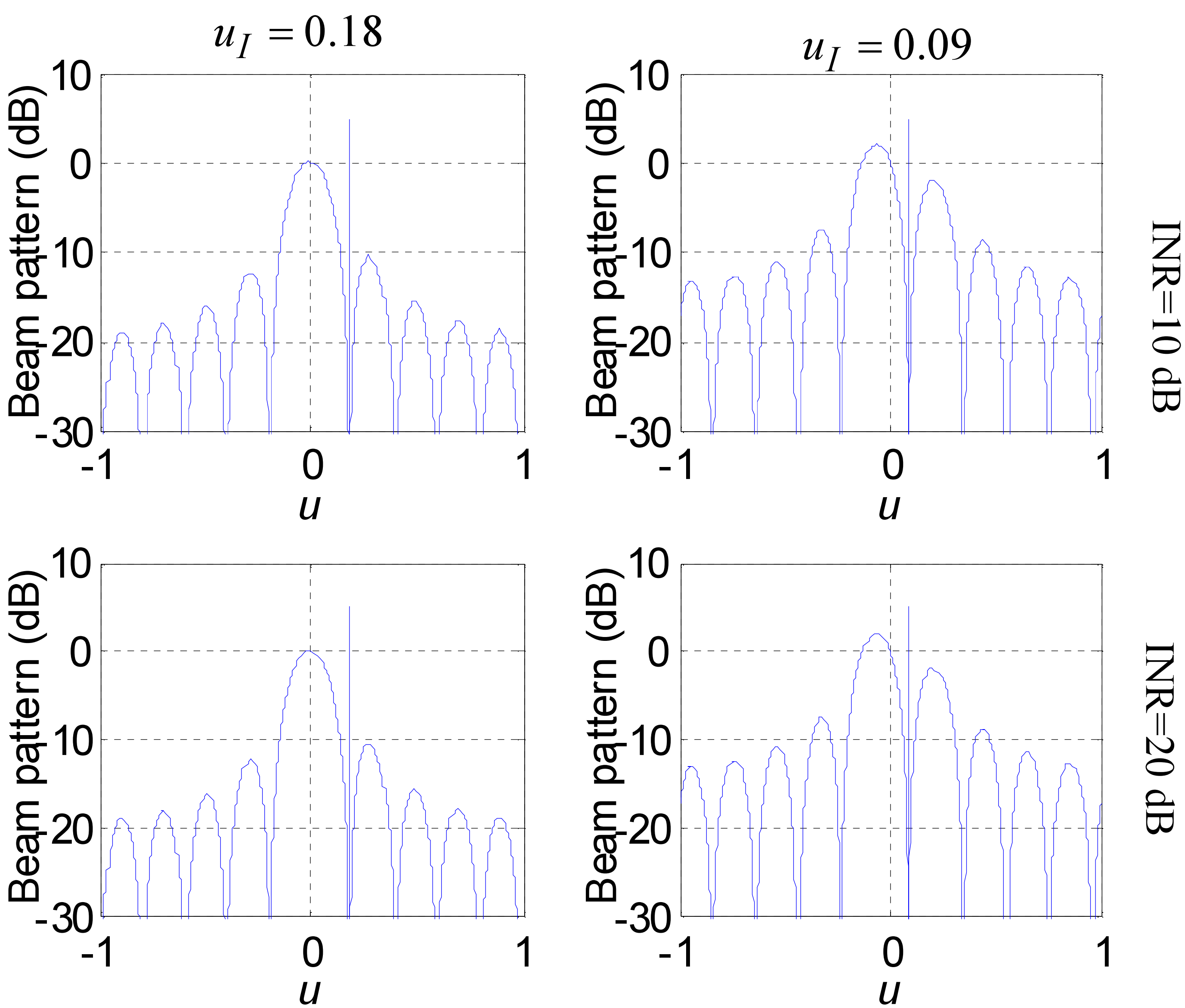
$$u = \cos\theta = \sin\bar{\theta}$$

Performance: Examples, main beam interferer



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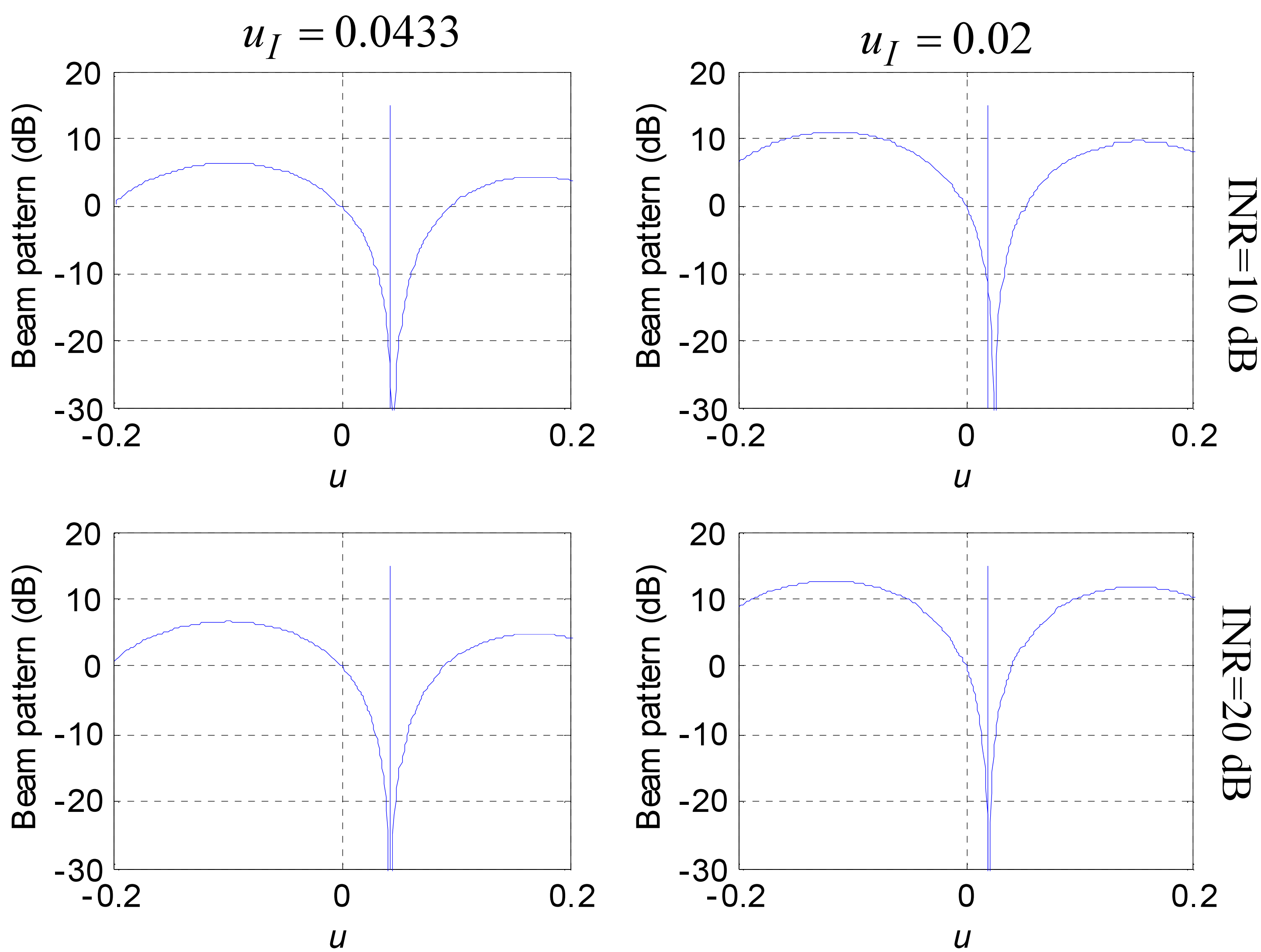
Performance: Examples, main beam interferer



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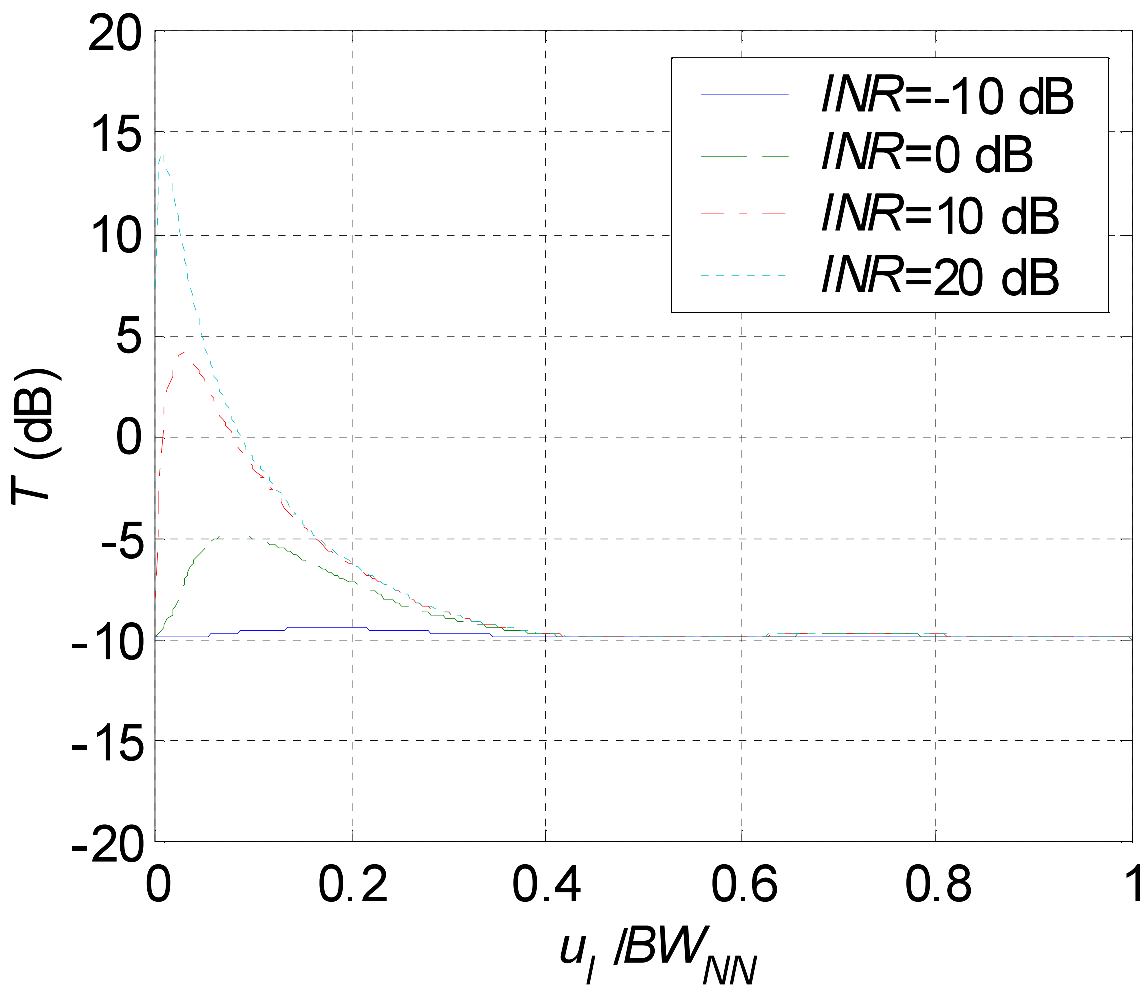
Performance: Sensitivity Function



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Explain the graph!

The array pattern is

$$F_0\left(\mathbf{k}/\mathbf{k}_s\right) = \frac{\gamma N}{\sigma_0^2}\left[F_c(\mathbf{k}/\mathbf{k}_s) - \frac{N\sigma_1^2}{\sigma_0^2 + N\sigma_1^2} F_c(\mathbf{k}/\mathbf{k}_1) F_c(\mathbf{k}_1/\mathbf{k}_s) \right] \tag{8.24}$$

The 1st term is a conventional pattern steered at $\mathbf{k}_s$, the 2nd term is a conventional pattern steered at $\mathbf{k}_1$(interferer) times conv. pattern steered at $\mathbf{k}_s$ and evaluated at $\mathbf{k}_1$. The total pattern is a difference of two.

Note that for high INR, $\alpha \gg 1$,

$$F_0(\mathbf{k}) \approx \frac{\gamma N}{\sigma_0^2}\left[F_c(\mathbf{k}/\mathbf{k}_s) - F_c(\mathbf{k}/\mathbf{k}_1) \cdot F_c(\mathbf{k}_1/\mathbf{k}_s) \right] \tag{8.25}$$

i.e. perfect null at the interferer direction, $F_0(\mathbf{k}_1) = 0$

Performance analysis

Performance is best when the interfere is outside of the main beam. In this case, $G_0 \approx N\alpha$ and $SNR_{out} = \alpha N \cdot SNR_{in}$, the array gain is large. The pattern is almost the same as conventional , except for the region close to the null.

Performance Analysis

- When the interfere approaches the main beam, there are two effects:

1) The main beam is shifted away from the null, its height is larger than unity.

2) The height of the sidelobe (closest to the null) increases (to about a few dBs). Performance is worse than above.

- When the interferer is inside of the main beam, the beam splits in two "side lobes" whose heights are large (> signal direction heights). The peak of the pattern is no longer pointed at the signal.

This is bad -> may result in increased noise from those directions, as well as in increased interference if there is any at that region. In this case, the beamformer is sensitive to the signal AOA mismatch.

Sometimes, this is acceptable, but in many cases we need to impose additional constraint to decrease this sensitivity.

Consider sensitivity function

$$T_s(\mathbf{k}_s) = \left|\mathbf{w}^+(\mathbf{k}_s)\right|^2 \quad (8.26)$$

For MVDR in the single-interferer case , we obtain

$$T_s = \frac{1 + 2N\alpha(1-|r_{s1}|^2) + N^2\alpha^2(1-|r_{s1}|^2)}{N\left[1 + N\alpha(1-|r_{s1}|^2)\right]^2} \quad (8.27)$$

Consider the case of $N\alpha(1-|r_{s1}|^2) >> 1$ and $|r_{s1}| < 1$,

$$T_s \approx \frac{1}{N(1-|r_{s1}|^2)} \tag{8.28}$$

i.e. the sensitivity increases when the correlation increases.

Q: for $|r_{s1}| = 1$, $T = \frac{1}{N}$ Explain this!

Q: what is T when $N\alpha \ll 1$? Explain it.

Summary

- Maximum SNR and MPDR beamformers.
- Comparison of MVDR, MMSE, max. SNR and MPDR beamformers.
- Performance analysis. The optimum pattern and null formation.
- Examples of patters.
- Effect of the interferer AOA.

References

1. H.L. Van Trees, Optimum Array Processing, Wiley, 2002.
2. R.A. Monzingo, T.W. Miller, Introduction to Adaptive Arrays, Wiley, 1980 (or 2011), Ch. 3.1-3.3.
3. J.E. Hudson, Adaptive Array Principles, London, 1981.
4. J.C. Liberti, Jr., T.S. Rappaport, Smart Antennas for Wireless Communications, Prentice Hall, 1999.

Homework

Fill in the details in the derivations above. Do the examples yourself.