Space-Time Random Processes

Time-domain random process: x(t), x - random variable, t – parameter (time). For any value of t, x(t) = r.v.

Autocorrelation function:

$$R_x(\tau) = \left\langle x(t)x^*(t+\tau) \right\rangle \tag{6.1}$$

where < > is statistical expectation. Note that R_x depends on the time difference τ only as we assume wide sense stationary (WSS) process. Its power is $P_x = R_x(0) = \left\langle xx^* \right\rangle$.

In general (non-WSS process),

$$R_x(t_1, t_2) = \left\langle x(t_1)x^*(t_2) \right\rangle \tag{6.2}$$

but we will not use it, always making the WSS assumption.

For deterministic signal x(t) :

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t+\tau)dt \qquad \text{(E.T. signals)} \tag{6.3}$$

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x(t)x^*(t+\tau)dt \qquad \text{(P. T. signals)} \tag{6.4}$$

The signal energy or power can be compactly expressed as $R_x(0)$.

Further, we use < > for both statistical and deterministic average. Also, for most of our models, $\left\langle x(t) \right\rangle = 0$, i.e. no DC component.

Autocorrelation Function and Power Spectrum

Space time random process: $x(t,\mathbf{p})$, $\mathbf{p}$ - position vector.

Autocorrelation function (space-time)

$$R_x(t_1,t_2,\mathbf{p}_1,\mathbf{p}_2) = \left\langle x(t_1,\mathbf{p}_1)x^*(t_2,\mathbf{p}_2)\right\rangle$$
$$= \left\langle x(t_1,\mathbf{p}_1)x^*(t_1+\tau,\mathbf{p}_1+\Delta\mathbf{p})\right\rangle$$
$$= R_x(\tau,\Delta\mathbf{p}) \tag{6.5}$$

i.e. WSS is space and time, and assuming that $\left\langle x(t,\mathbf{p})\right\rangle = 0$ (if not, subtract the mean from x in (6.5)).

For deterministic $x(t,\mathbf{p})$: the same as for time only + space coordinate.

Power spectrum or PSD (ESD for E.T. signal) of a time-domain signal:

$$S_x(\omega) = FT\{R_x(\tau)\} = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau}d\tau \tag{6.6}$$

For deterministic x(t),

$$X(\omega) = FT\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \tag{6.7}$$

$$S_x(\omega) = FT\{R_x(\tau)\} = |X(\omega)|^2 \tag{6.8}$$

This justifies the term “power spectrum”.

Space-Time Power Spectrum and Correlation Matrix

Space-time power spectrum:

$$S_x(\omega, \Delta\mathbf{p}) = FT\left\{R_x(\tau, \Delta\mathbf{p})\right\}_\tau$$
$$= \int_{-\infty}^{\infty} R_x(\tau, \Delta\mathbf{p}) e^{-j\omega\tau} d\tau \qquad (6.9)$$

Correlation Matrix

Consider the vector signal $\mathbf{x}(t)$ (the array output):

$$\mathbf{x}(t) = \left[x_1(t) \quad x_2(t) \quad \ldots \quad x_N(t)\right]^T$$
$$= \left[x(t, \mathbf{p}_1) \quad x(t, \mathbf{p}_2) \quad \ldots \quad x(t, \mathbf{p}_N)\right]^T \qquad (6.10)$$

The (also “covariance” or “spectral”) matrix is defined as

$$\left[S_x(\omega)\right]_{ij} = S_x(\omega, \mathbf{p}_i - \mathbf{p}_j) \qquad (6.11)$$

In fact,

$$\mathbf{R}_x(\tau) = \left\langle \mathbf{x}(t)\mathbf{x}^+(t+\tau) \right\rangle \qquad (6.12)$$

is the correlation matrix, and

$$\mathbf{S}_x(\omega) = FT\left\{\mathbf{R}_x(\tau)\right\}_\tau \qquad (6.13)$$

This is a vector (array) analog of the power spectrum. *FT* is applied to each component of $\mathbf{R}_x$ individually.

The correlation matrix $\mathbf{S}_x$ will play a central role in beamforming algorithms.

Its component-wise representation is:

$$\left[\mathbf{R}_x(\tau)\right]_{ij} = \left\langle x_i(t) x_j^*(t+\tau) \right\rangle \tag{6.14}$$

This is cross-correlation (or, simply, correlation) between signals of various array elements.

Narrow-band signals

Consider the complex envelope representation of a narrowband signal: $x_i(t) = A_i(t)e^{j\omega_c t}$, where $A_i(t)$ is a complex amplitude (varies much more slowly than the carrier term $e^{j\omega_c t}$).

The signal of i-th array element is delayed by

$$x_i(t) = x(t-\tau_i) = A(t-\tau_i)e^{j\omega_c(t-\tau_i)} \tag{6.15}$$

For narrowband signals,

$$A(t-\tau_i) = A(t), \quad x_i(t) = A(t)e^{j\omega_c t}e^{-j\omega_c \tau_i} \tag{6.16}$$

The effect of delay is to introduce a phase shift $e^{-j\omega_c \tau_i}$ only.

The received narrowband signal can be expressed as

$$\mathbf{x}(t) = x(t)\mathbf{v}(\mathbf{k}) \tag{6.17}$$

where $x(t) = A(t)e^{j\omega_c t}$ is the plane-wave waveform coming from the direction $\mathbf{k}$, and the array manifold vector

$$\mathbf{v}(\mathbf{k}) = \left[e^{-j\mathbf{k}\mathbf{p}_1}, e^{-j\mathbf{k}\mathbf{p}_2}, \ldots e^{-j\mathbf{k}\mathbf{p}_N}\right]^T \tag{6.18}$$

In the frequency domain:

$$\mathbf{x}(\omega) = x(\omega)\cdot\mathbf{v}(\mathbf{k}) \tag{6.19}$$

where $\mathbf{x}(\omega) = FT\{\mathbf{x}(t)\}, \quad x(\omega) = FT\{x(t)\}$.

Q.: verify that (6.17) indeed represents the array response to a plane wave.

This may be generalized to broad-band signals as well. However, in most cases we will consider narrow-band signals only and will suppress ω.

In fact, different elements receive phase-shifted copies of the same spectrum. We drop ω below assuming flat spectrum for simplicity (alternatively, x and ξ can be considered as complex envelopes).

Example 1: narrowband plane wave + noise,

$$\mathbf{x} = \mathbf{x}_s + \boldsymbol{\xi} \tag{6.20}$$

where $\mathbf{x}_s = x_s \mathbf{v}(\mathbf{k})$ is a plane-wave signal, $\boldsymbol{\xi}$ is the (thermal) noise.

Assume that the signal and noise are uncorrelated.

Correlation matrix (power spectrum):

$$\mathbf{S}_x = \left\langle \mathbf{x}\mathbf{x}^+ \right\rangle = \mathbf{S}_s + \mathbf{S}_\xi \tag{6.21}$$

$$\mathbf{S}_s = \left\langle \mathbf{x}_s \mathbf{x}_s^{\;+} \right\rangle = \sigma_s^2 \mathbf{v}(\mathbf{k}) \mathbf{v}^+(\mathbf{k}) \tag{6.22}$$

$$\mathbf{S}_\xi(\omega) = \sigma_\xi^2 \mathbf{I} \tag{6.23}$$

where $\sigma_s^2 = \left\langle x_s x_s^{\;*} \right\rangle$ is the power of x_s [signal's complex envelope]; the noise is assumed to be uncorrelated at different elements, i.e.

$$\left\langle \xi_i \xi_j^* \right\rangle = 0, \quad i \neq j \tag{6.24}$$

and

$$\sigma_\xi^2 = \left\langle \xi_i \xi_i^* \right\rangle \tag{6.25}$$

is the noise variance (power), assumed to be the same in all elements.

We will use this model frequently in developing beamforming algorithms. Make sure you understand it well!

Q.: derive (6.21)-(6.23). Hint: you may assume that both the noise and the signal are white, i.e. their autocorrelation function is $R(\tau) = \sigma^2 \delta(\tau)$, where $\delta(\tau)$ is the Dirac delta function.

The noise with $\mathbf{S}_\xi = \sigma_\xi^2 \mathbf{I}$ is called "spatial white noise".

<u>Example 2:</u> *M* plane-wave signals + noise

$$\mathbf{x} = \sum_{i=1}^{M} \mathbf{x}_{s,i} + \boldsymbol{\xi} \tag{6.26}$$

where $\mathbf{x}_{s,i} = x_{s,i} \mathbf{v}(\mathbf{k}_i)$, and the complex envelope of the i-th incoming plane-wave signal is $x_{s,i}$. The correlation matrix is

$$\mathbf{S}_x = \mathbf{V}_s \mathbf{S}_s \mathbf{V}_s^+ + \sigma_\xi^2 \mathbf{I} \tag{6.27}$$

$$\mathbf{V}_s = \left[\mathbf{v}(\mathbf{k}_1) \quad \mathbf{v}(\mathbf{k}_2) \quad \ldots \quad \mathbf{v}(\mathbf{k}_M) \right] \tag{6.28}$$

$$\left[\mathbf{S}_s \right]_{ij} = \left\langle x_{s,i} x_{s,j}^* \right\rangle \tag{6.29}$$

where $\mathbf{V}_s$ is the array manifold matrix. The noise correlation matrix is $\mathbf{S}_\xi = \sigma_\xi^2 \mathbf{I}$.

Q.: derive (6.27). Consider the case of uncorrelated signal's envelopes, and compare it to the case of 1 plane-wave signal (6.20).

Q.: prove that any correlation matrix $\mathbf{S}$ has the following properties:

1. $\mathbf{S}$ is Hermitian, i.e. $\mathbf{S} = \mathbf{S}^+$.
2. $\mathbf{S}$ is positive semi-definite, i.e. $\mathbf{x}^+ \mathbf{S} \mathbf{x} \geq 0 \forall \mathbf{x}$.
3. $\mathbf{S}$ has non-negative eigenvalues.

Q.: prove the following properties of $\mathbf{S}_x$ above:

1. $\mathbf{S}_x$ is positive definite, i.e. $\mathbf{x}^+ \mathbf{S} \mathbf{x} > 0 \forall \mathbf{x} \neq \mathbf{0}$.
2. $\mathbf{S}_x$ has positive eigenvalues.

We will use below the example 1 and 2 models to design optimum beamformers. Required signal and interference will be considered to be plane waves.

Summary

- Characterization of regular (time-domain) and space-time random processes.
- Power (energy) spectrum.
- Correlation (spectral) matrices.
- Spatially-white noise. Narrowband processes.
- Plane-wave signals.

References:

1. H.L. Van Trees, Optimum Array Processing, Wiley, New York, 2002.
2. R.A. Monzingo, T.W. Miller, Introduction to Adaptive Arrays, Wiley, New York, 1980, Ch. 3.
3. J.E. Hudson, Adaptive Array Principles, Peter Peregrinus, London, 1981.
4. J.C. Liberti, Jr., T.S. Rappaport, Smart Antennas for Wireless Communications, Prentice Hall, Upper Saddle River, 1999.

Note that some books are available in pdf files: check the library catalogue!

Homework

Fill in the details in the derivations above. Do the examples yourself.