Signal Space

Geometrical interpretation of signals via orthogonal basis function expansion:

$$s(t) = \sum_{i=1}^{N} s_i \psi_i(t), \quad 0 \le t \le T, \quad s(t) \leftrightarrow \{s_i\} \Rightarrow \mathbf{s} \qquad (9.1)$$

i.e. each signal $s(t)$ is represented by vector $\mathbf{s}$ of expansion coefficients $\{s_i\}$.

$\{\psi_i(t)\}$ are a set of orthonormal basis functions,

$$\int_T \psi_i \psi_j^*(t) dt = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} = \delta_{ij}$$

$$E_\psi = \int_T \psi^2(t) dt = 1 \qquad (9.2)$$

Linear independence of $\{\psi_i(t)\}$:

$$\sum_{i=1}^{N} \alpha_i \psi_i(t) = 0 \ \ \forall t \in [0,\mathrm{T}] \rightarrow \alpha_i = 0 \ \ \forall i \qquad (9.3)$$

Any complete set of LI functions can be used in (9.1), but orthonormal is more convenient.

Any LI set $\Rightarrow$ orthonormal set via Gram-Schmidt orthogonalization process.

(9.3) $\Rightarrow$ same as for linear independence of vectors.

Example: Fourier series

Consider periodic signal $s(t+T)=s(t)$ $\forall$t,

$$s(t)=\sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi}{T}kt}, \quad c_k=\frac{1}{T}\int_T s(t)e^{-j\frac{2\pi}{T}kt}dt \qquad (9.4)$$

Can take $\psi_k(t)=\frac{1}{\sqrt{T}}e^{j\omega t}$, $\omega=\frac{2\pi}{T}=$ fundamental frequency,

$$\int_T \psi_k(t)\psi_n^*(t)dt=\delta_{kn} \qquad (9.5)$$

Complex vs. real form of FS: $c_k \leftrightarrow \{a_k, b_k\}$

$$s(t)=\sum_{k=0}^{\infty} a_k \cos(k\omega t)+b_k \sin(k\omega t) \qquad (9.6)$$

Can take $\psi_{1,k}(t)=\sqrt{\frac{2}{T}}\cos(k\omega t)$, $\psi_{2,k}(t)=\sqrt{\frac{2}{T}}\sin(k\omega t)$.

Example: M-PAM

$$s_i(t)=A_i p(t), \quad i=1...M, \quad 0\le t\le T \qquad (9.7)$$

Take $\psi(t)=\frac{1}{\sqrt{E_p}}p(t)$.

Example: QAM

I – Q representation of bandpass signals:

$$x(t) = m_I(t)\cos\omega t + m_Q(t)\sin\omega t \tag{9.8}$$

Take $\psi_1(t) = \sqrt{\frac{2}{T}}\cos\omega t, \quad \psi_2(t) = \sqrt{\frac{2}{T}}\sin\omega t$.

These basis functions are very important and are used often in communications.

How to find $\{s_i\}$?

$$s_i = \int_T s(t)\psi_i^*(t)dt \leftrightarrow s(t) = \sum_i s_i\psi_i(t) \tag{9.9}$$

Signal energy: ("distance" from 0)

$$E_s = \int_T |s(t)|^2 dt = |\mathbf{s}|^2 = \sum_i |s_i|^2 \tag{9.10}$$

Scalar product of $x(t)$ and $y(t)$:

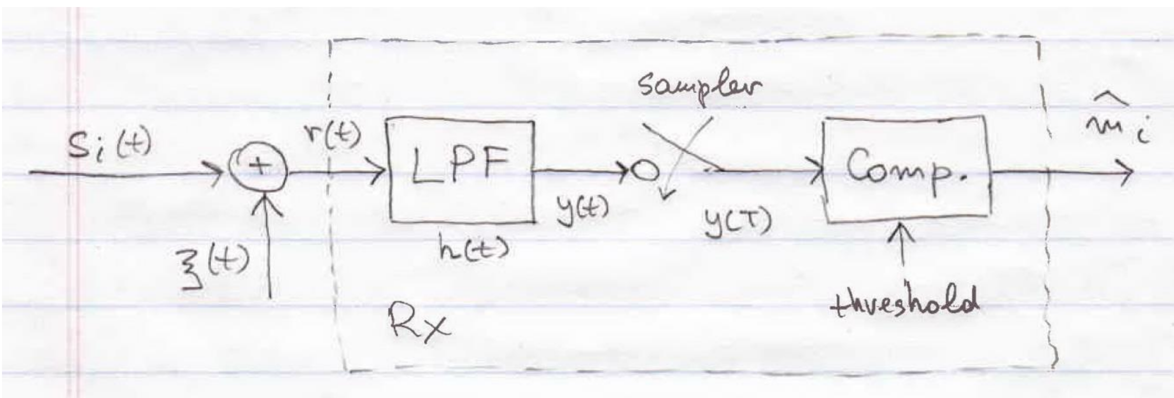
$$\mathbf{x}^+\mathbf{y} = \sum_i x_i^* y_i = \int_T x^*(t)y(t)dt \tag{9.11}$$

<u>Distance</u> between $x(t)$ and $y(t)$:

$$|\mathbf{x}-\mathbf{y}|^2 = \sum_i |x_i - y_i|^2 = \int_T |x(t)-y(t)|^2\, dt \qquad (9.12)$$

$\{\psi_i(t)\}$ = basis "vectors" in the signal space.

<u>Optimal Rx structure (Lec. 7):</u>



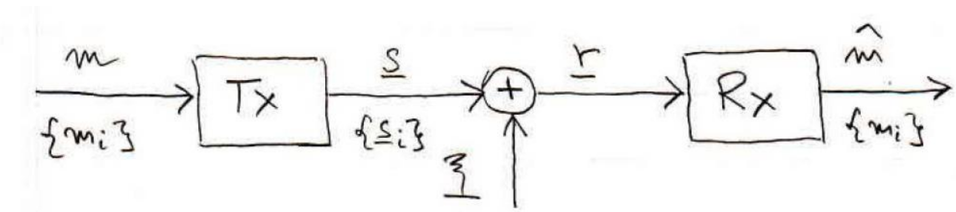
* Optimality was not proved.

* Will be proved below via probabilistic analysis in the signal space.

Optimal Rx in Signal Space (AWGN)

Key idea: replace $r(t)$ by $\mathbf{r}$ and detect it. No loss of optimality (can be proved).

$$r(t) = s(t) + \xi(t) \;\leftrightarrow\; \mathbf{r} = \mathbf{s} + \boldsymbol{\xi} \tag{9.13}$$



$m =$ estimated message (Rx)
$m =$ selected message (S)
$\mathbf{s} =$ transmitted signal
$\mathbf{r} =$ received signal

Assume $\xi(t) =$ AWGN, then

$$\begin{aligned} &R(\tau) = \overline{\xi(t)\xi^*(t+\tau)} = \frac{N_0}{2}\delta(\tau), \;\; \overline{\xi(t)} = 0, \\ &S_\xi(f) = N_0 = \text{PSD} \end{aligned} \tag{9.14}$$

and

$$\begin{aligned} &\boldsymbol{\xi} \sim N(0, \sigma_0^2 \mathbf{I}), \quad \sigma_0^2 = \frac{N_0}{2} = \text{var}(\xi_i) \\ &p_\xi(\boldsymbol{\xi}) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{N_0} \right) \end{aligned} \tag{9.15}$$

Also $\mathbf{r} \sim N(\mathbf{s}, \sigma_0^2 \mathbf{I})$ for a given $\mathbf{s}$.

Transmitter: $m \to \mathbf{s}; \quad m_i \in \{m_1 ... m_M\} = \{m_i\}$

$$\mathbf{s} \in \{\mathbf{s}_1 ... \mathbf{s}_M\} = \{\mathbf{s}_i\}$$

Receiver: $\mathbf{r} \to \hat{\mathbf{s}} \to m$

Optimal Rx: $P_e \to \min$,

$$P_e = \sum_{i=1}^{M} \Pr\{m_i\} P_{e_i}, \quad P_{e_i} = \{\hat{\mathbf{s}} \neq \mathbf{s}_i\} \qquad (9.16)$$

Consider the equiprobable messages (why important?):

$$\Pr\{m_k\} = \frac{1}{M} \Rightarrow \boxed{m = m_k \text{ if } |\mathbf{r} - \mathbf{s}_k| \leq |\mathbf{r} - \mathbf{s}_i| \ \forall i \neq k} \qquad (9.17)$$

i.e. the maximum-likelihood (ML) decision rule. Can also be used when $\Pr\{m_k\}$ are not known.

$$\boxed{\mathrm{ML} = \min.\ \mathrm{distance}} \text{ in AWGN} \qquad (9.18)$$

<u>Decision region</u>:

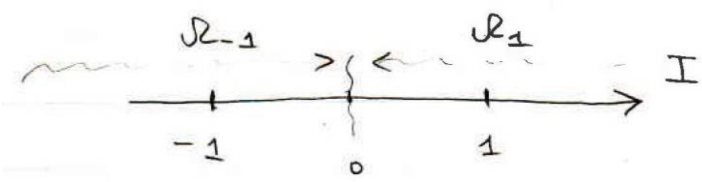
$$\Omega_k = \{\mathbf{r} : |\mathbf{r} - \mathbf{s}_k| \le |\mathbf{r} - \mathbf{s}_i| \ \forall i \ne k\} \tag{9.19}$$

so that the ML rule is

$$\boxed{m = m_k \ \text{if} \ \mathbf{r} \in \Omega_k} \tag{9.20}$$

$\{\Omega_i\}_{i=1}^{M}$: split all space into a set of disjoint sets/decisions regions.

<u>Example: BPSK</u>, $m_i = \pm 1, \ \psi(t) = \alpha p(t)$.



$m = 1 \ \text{if} \ r \in \Omega_1 \Rightarrow r > 0$

$m = -1 \ \text{if} \ r \in \Omega_{-1} \Rightarrow r < 0$

Example: QPSK

$$\psi_1(t) = \sqrt{\frac{2}{T}} \cos \omega t$$

$$\psi_2(t) = \sqrt{\frac{2}{T}} \sin \omega t$$

$$\mathbf{s}_1 = [1,1]^T \qquad \mathbf{s}_3 = [-1,-1]^T$$

$$\mathbf{s}_2 = [-1,1]^T \qquad \mathbf{s}_4 = [1,-1]^T$$

$$\Omega_1 = ? \qquad \Omega_{2,3,4} = ?$$

Signal constellation = $\{s_i(t)\}$ in the signal space, i.e. $\{\mathbf{s}_i\}$.

Signals: Points (vectors) in the signal space.

Receiver Implementation

The rule (9.17) can be expressed in a different form using

$$|\mathbf{r}-\mathbf{s}_k|^2 = |\mathbf{r}|^2 - 2\mathbf{r}\mathbf{s}_k + |\mathbf{s}_k|^2 \qquad (9.21)$$

Since $|\mathbf{r}|^2$ is independent of $\mathbf{s}$, it can be dropped and (9.17) becomes

$$\mathbf{r}^+\mathbf{s}_k + c_k \ge \mathbf{r}^+\mathbf{s}_i + c_i, \forall i \ne k \qquad (9.22)$$

where $\mathbf{r}^+\mathbf{s}_k = \sum_{i=1}^{N} r_i s_{ki}$ is a scalar product and $c_k = -|\mathbf{s}_k|^2/2$. It can be expressed as

$$\mathbf{r}^+\mathbf{s}_k = \int_T r(t)s_k(t)dt \qquad (9.23)$$

Q. Prove it.

(9.23) can be implemented using a correlation receiver or a matched filter receiver.

Matched filter impulse response

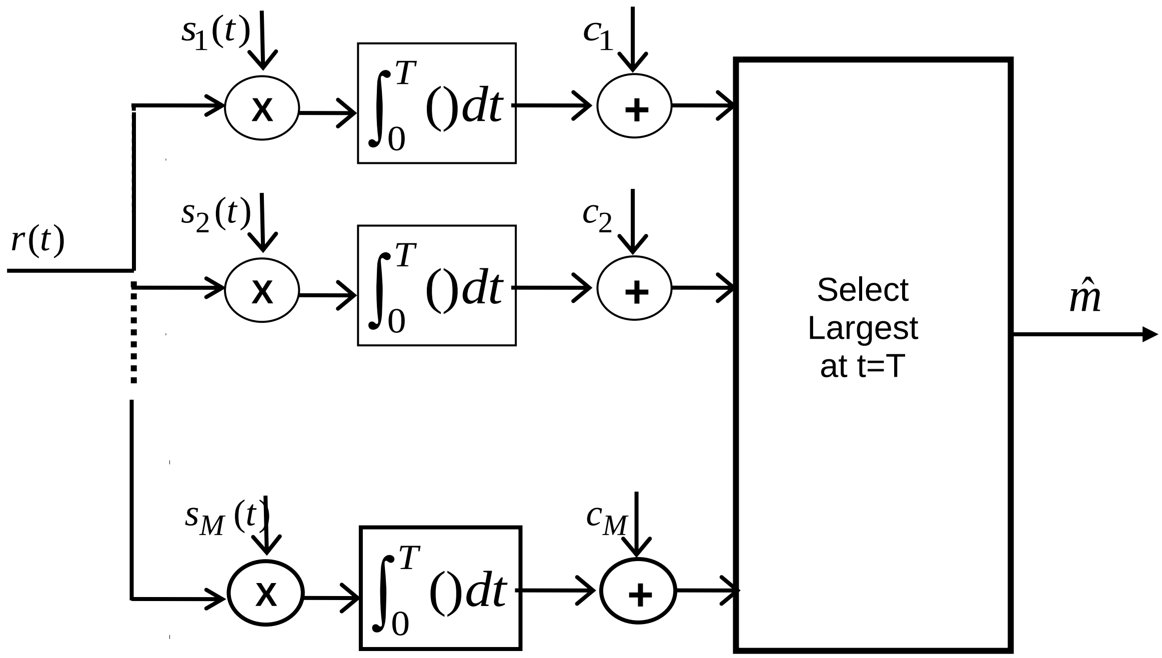
$$h_k(t) = s_k(T-t) \qquad (9.24)$$

so the output sampled at time T is

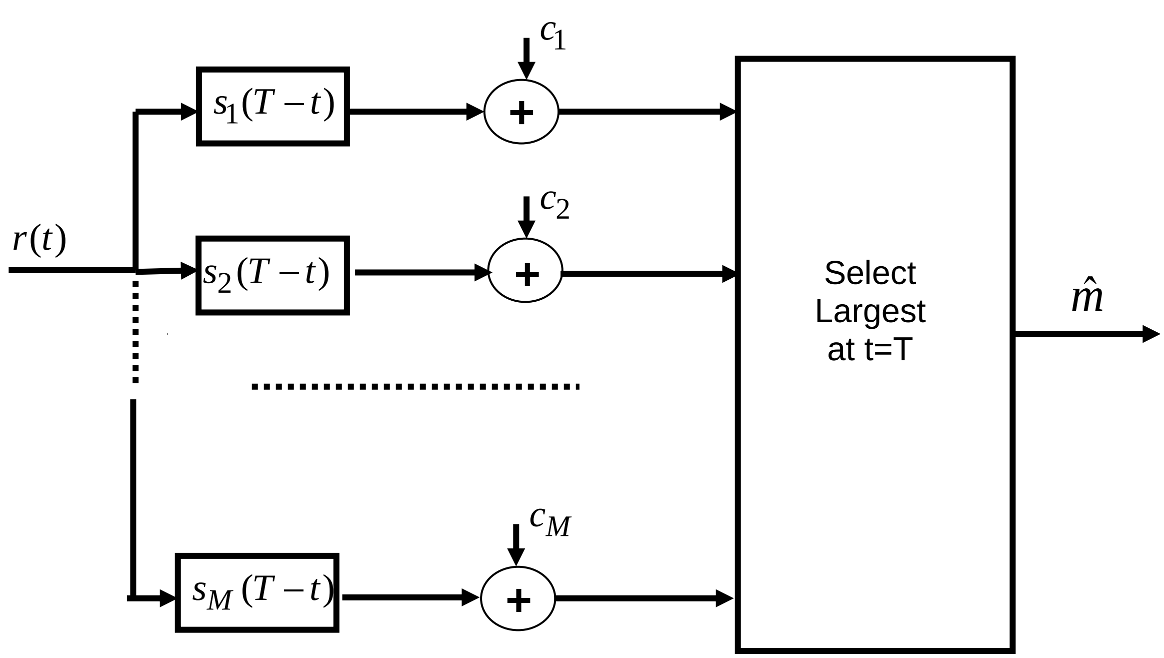
$$\rho(t) * h_k(t) = \int_T r(t)s_k(t)dt \qquad (9.25)$$

Implementation

Correlation Receiver

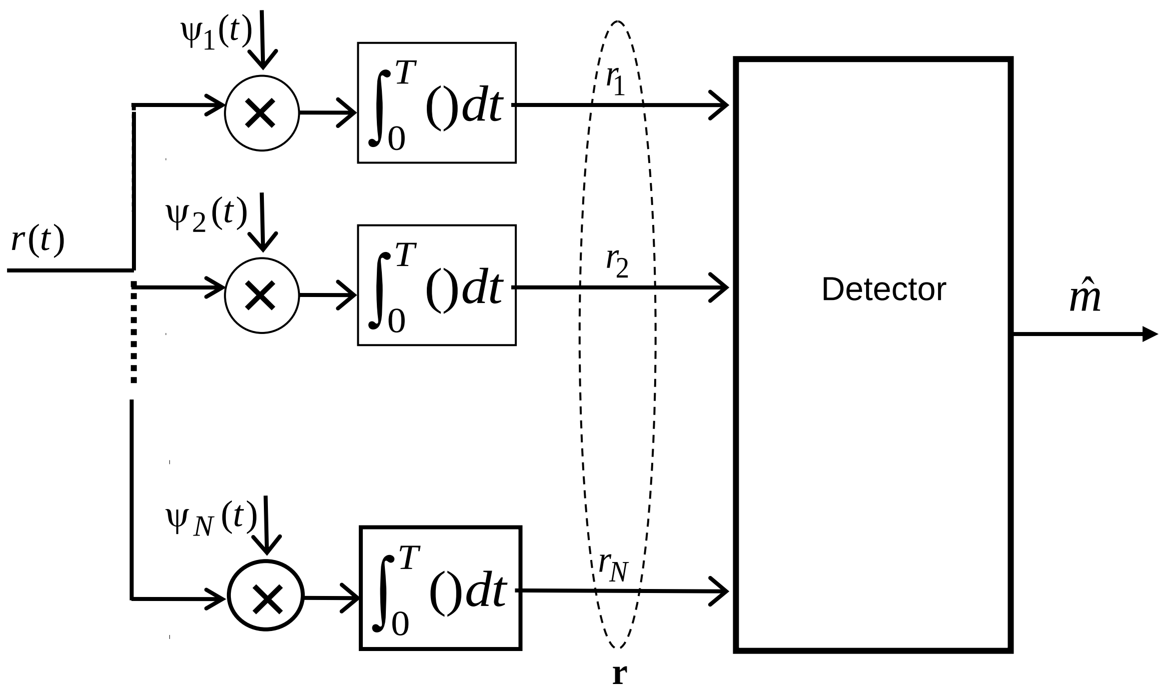


Matched Filter Receiver



Another Form of Implementation

Correlation receiver in the signal space



Q.: detector =?

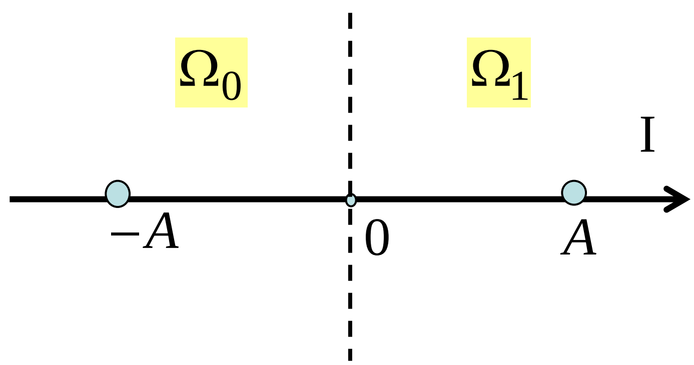
An Example: BPSK

Time-domain expression of the signal is

$$s_i(t) = (-1)^i A \cdot \psi(t), \quad i = 0,1 \tag{9.26}$$

where $\psi(t)$ is a basis pulse shape (may be $\cos(\omega t)$); A is the amplitude.

Constellation:



The ML receive computes the following decision variable,

$$z = \int_T r(t)\psi(t)dt \tag{9.27}$$

and sets

$$\hat{m} = \begin{cases} 0, & if \quad z < 0 \\ 1, & if \quad z > 0 \end{cases}$$

Q.: block diagram of the receiver?

Comparison of M-ary Modulation Schemes

M-PSK: the phase can assume M different values

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_c t + \frac{2\pi}{M} i\right),\ i = 0,1,\ldots,M-1 \qquad (9.28)$$

$$0 \le t \le T$$

$\log_2 M$ bits are transmitted by each symbol.

The symbol error probability (SER):

$$P_{se} \approx 2Q\left(\sqrt{\frac{2E}{N_0}} \sin\left(\frac{\pi}{M}\right)\right) = \alpha Q\left(\sqrt{\frac{\beta E}{N_0}}\right) \qquad (9.29)$$

Q: constellation example? Minimum distance $d_{\min}$?

Table 6.4 Bandwidth and Power Efficiency of M-ary PSK Signals

M	2	4	8	16	32	64
$\eta_B = R_b/B$ *	0.5	1	1.5	2	2.5	3
E_b/N_o for BER=10^{-6}	10.5	10.5	14	18.5	23.4	28.5

* B: First null bandwidth of M-ary PSK signals

T.S. Rappaport, Wireless Communications, Prentice Hall, 2002

*rectangular pulse or RC pulse with $\alpha = 1$

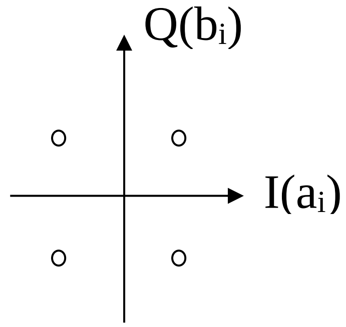
see (9.45) for the power/bandwidth efficiency tradeoff.

M-ary Quadrature AM (M-QAM)

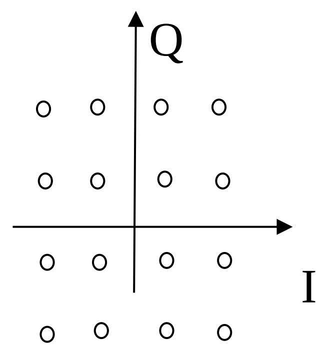
M levels with different phases and amplitudes

$$s_i(t) = a_i \cos \omega_c t + b_i \sin \omega_c t \tag{9.30}$$

a_i and b_i - I and Q components.



M=4



M=16

The BER of M-QAM ($M = 2^k$ and k is even):

$$P_e \approx \frac{4\left(1 - 1/\sqrt{M}\right)}{\log_2 M} Q\left[\sqrt{\frac{3\log_2 M}{M-1} \cdot \frac{E_b}{N_0}}\right] \tag{9.31}$$

Table 6.5 Bandwidth and Power Efficiency of QAM [Zie92]

M	4	16	64	256	1024	4096
η_B	1	2	3	4	5	6
E_b/N_o for BER = 10^{-6}	10.5	15	18.5	24	28	33.5

T.S. Rappaport, Wireless Communications, Prentice Hall, 2002

The threshold SNR: $\gamma_{th} \sim 10 \lg M$

Frequency Shift Keying (FSK)

M-ary FSK:

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \varphi),\ \mathrm{i} = 1, \ldots, M \quad 0 \le t \le T \qquad (9.32)$$

Note: signal orhtogonality imposes a limit on $\Delta\omega = \omega_{i+1} - \omega_i$.

Q: find min $\Delta\omega$ such that the signals are orthogonal.

For orthogonal BFSK (coherently detected),

$$P_e = Q\left(\sqrt{\gamma}\right) \qquad (9.33)$$

Note: non-coherent detection results in different BER,

$$P_{e,N} = e^{-\gamma/2}/2 \qquad (9.34)$$

Performance loss is a few dBs.

For $M > 2$, the tight upper bound on SER is

$$P_{es} \le (M-1)Q\left(\sqrt{\gamma}\right) \qquad (9.35)$$

for orthogonal signals and coherent demodulation.

Table 6.6 Bandwidth and Power Efficiency of Coherent M-ary FSK [Zie92]

M	2	4	8	16	32	64
η_B	0.4	0.57	0.55	0.42	0.29	0.18
E_b/N_o for BER = 10^{-6}	13.5	10.8	9.3	8.2	7.5	6.9

T.S. Rappaport, Wireless Communications, Prentice Hall, 2002

SNR for Analog and Digital Systems

SNR in an analog systems is

$$SNR_a = \frac{S}{N}, \tag{9.36}$$

where S is the signal power, N is the noise power.
Note that $S = E / T_s$ and $R_s = 1/T_s$. Furthermore,
$N = N_0 \cdot \Delta f$. Hence

$$\gamma = \frac{E_s}{N_0} = \frac{S \cdot T_s}{N / \Delta f} = \frac{S}{N} \cdot \frac{\Delta f}{R_s} = SNR_a \frac{\Delta f}{R_s} \tag{9.37}$$

For $R_s = \Delta f$, they are the same. $\frac{\Delta f}{R_s}$ is inverse of the bandwidth efficiency, $R_s / \Delta f \quad (= \eta)$.
Similarly,

$$\gamma_b = \frac{E_b}{N_0} = SNR_a \frac{\Delta f}{R} \tag{9.38}$$

where R = bit rate (bit/s).

SER and Signal Constellation

SER can be expressed through minimum distance between points of the signal constellation.

Generic bound for the SER is (coherent detection),

$$P_e \le (M-1)Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) \qquad (9.39)$$

where $d_{\min}$ is determined from signal constellation (minimum distance), see the next slide.

An approximation at high SNR:

$$P_e \approx N_e Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right) \qquad (9.40)$$

where N_e =# of nearest neighbours.

The BER can be approximated at high SNR as

$$P_b \approx \frac{1}{\log_2 M} P_e \qquad (9.41)$$

Q.: what is an interpretation of (9.41)?

Comparison of Various Modulation Formats

Fundamental limit is provided by the Shannon's channel capacity theorem (AWGN channel):

$$\boxed{C = \Delta f \log_2 (1+\gamma)} \tag{9.42}$$

C = channel capacity [bit/s]

Δf = bandwidth [Hz]

γ = SNR, $\gamma = P/N$, where P - signal power, N - noise power.

Almost error-free transmission is possible if $R < C$, and is not possible for $R > C$; R =bit rate [b/s].

Using $E_b = PT_b$, $\gamma_b = \dfrac{E_b}{N_0} = \gamma \dfrac{\Delta f}{R}$,

$$\frac{C}{\Delta f} = \log_2 (1 + \gamma_b R / \Delta f) \tag{9.43}$$

Since $R < C$ for reliable communications, then

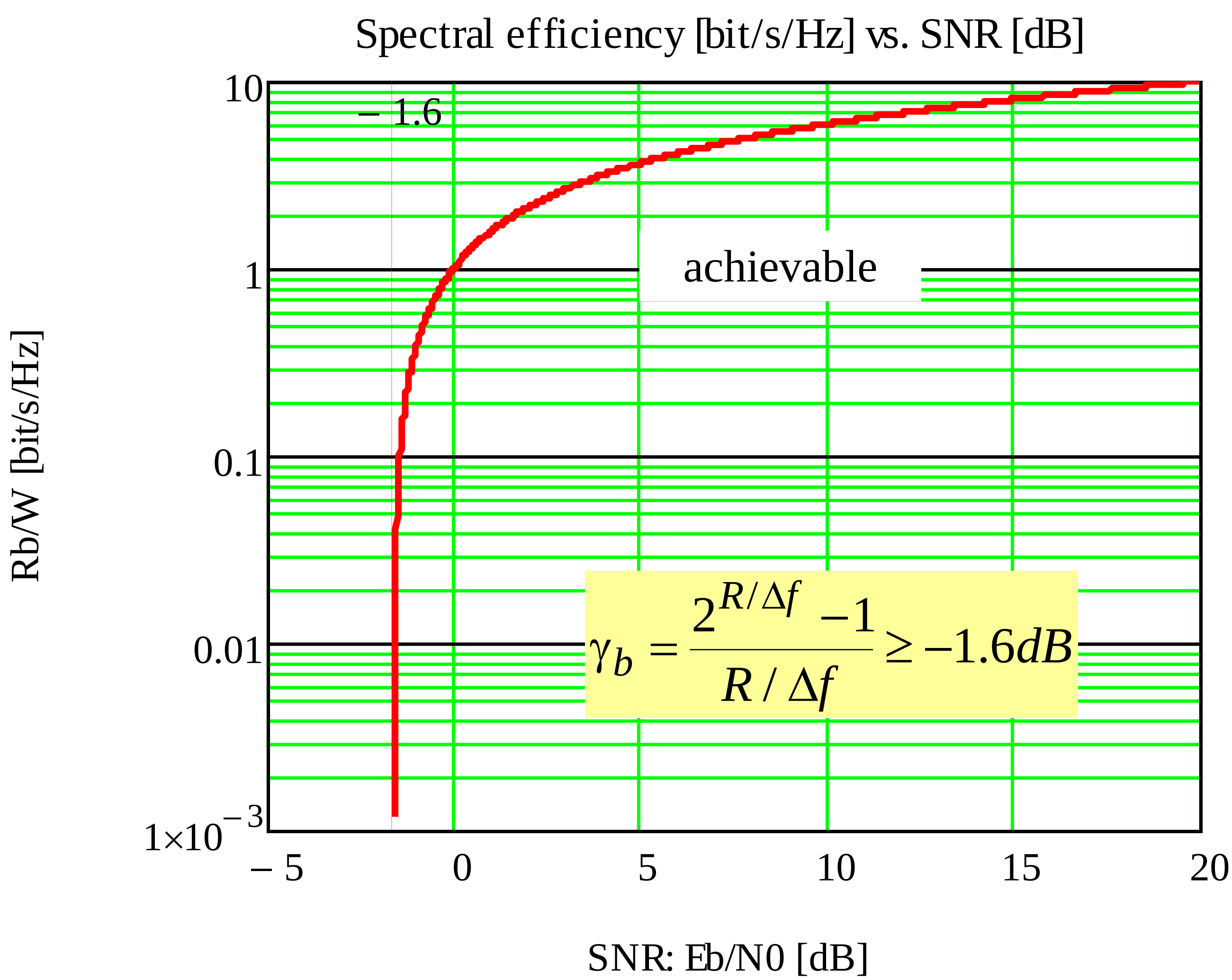
$$\frac{R}{\Delta f} < \log_2 (1 + \gamma_b R / \Delta f) \tag{9.44}$$

$R / \Delta f$ (bit/s/Hz) is the spectral efficiency. Required SNR is

$$\gamma_b > \frac{2^{R/\Delta f} - 1}{R / \Delta f} \geq \ln 2 = -1.6 dB \tag{9.45}$$

LB is monotonically increasing in $R / \Delta f$ - power/bandwidth efficiency tradeoff (see the tables).

Fundamental Limit: Spectral Efficiency [bit/s/Hz] vs. SNR/bit [dB]



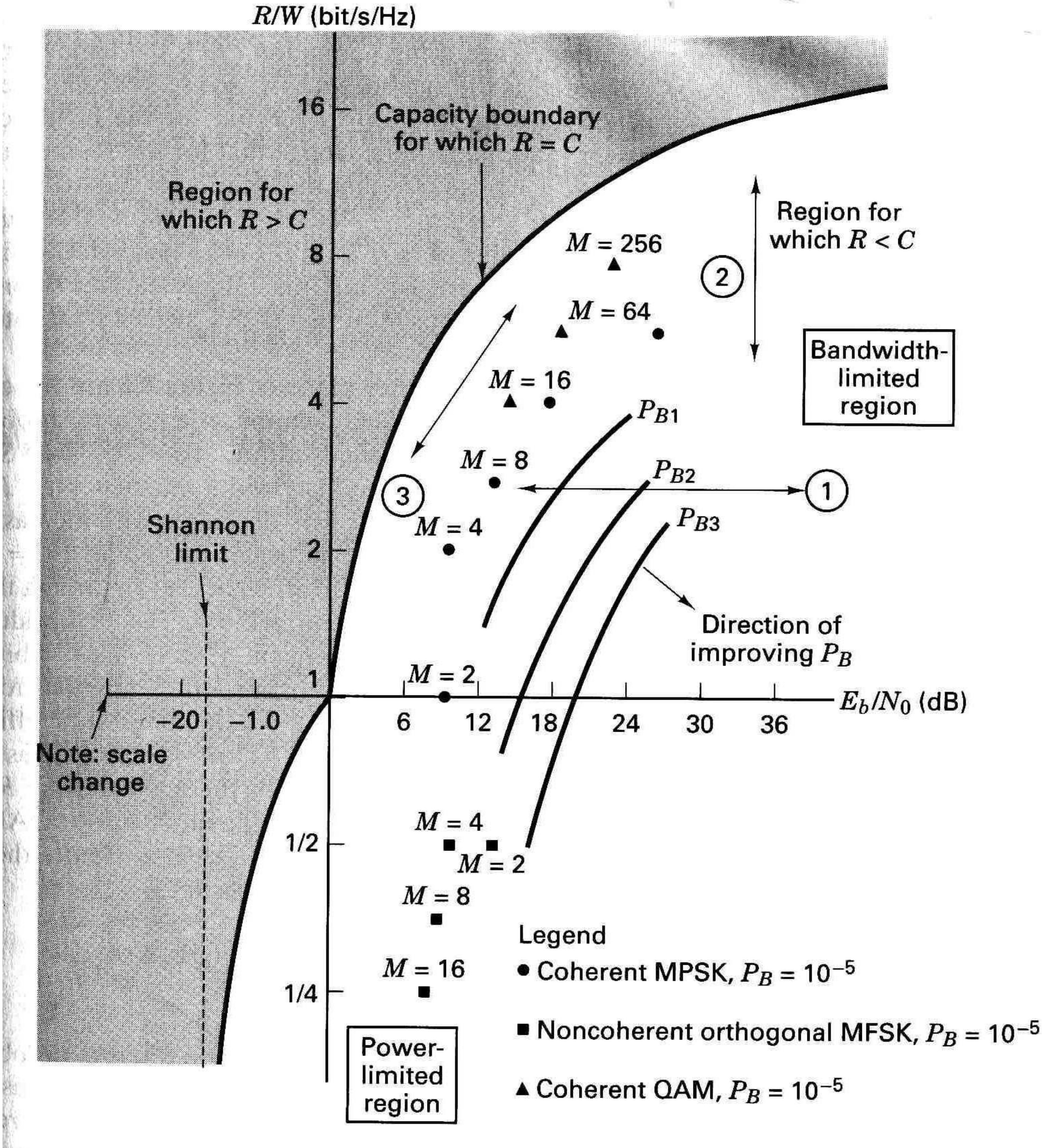
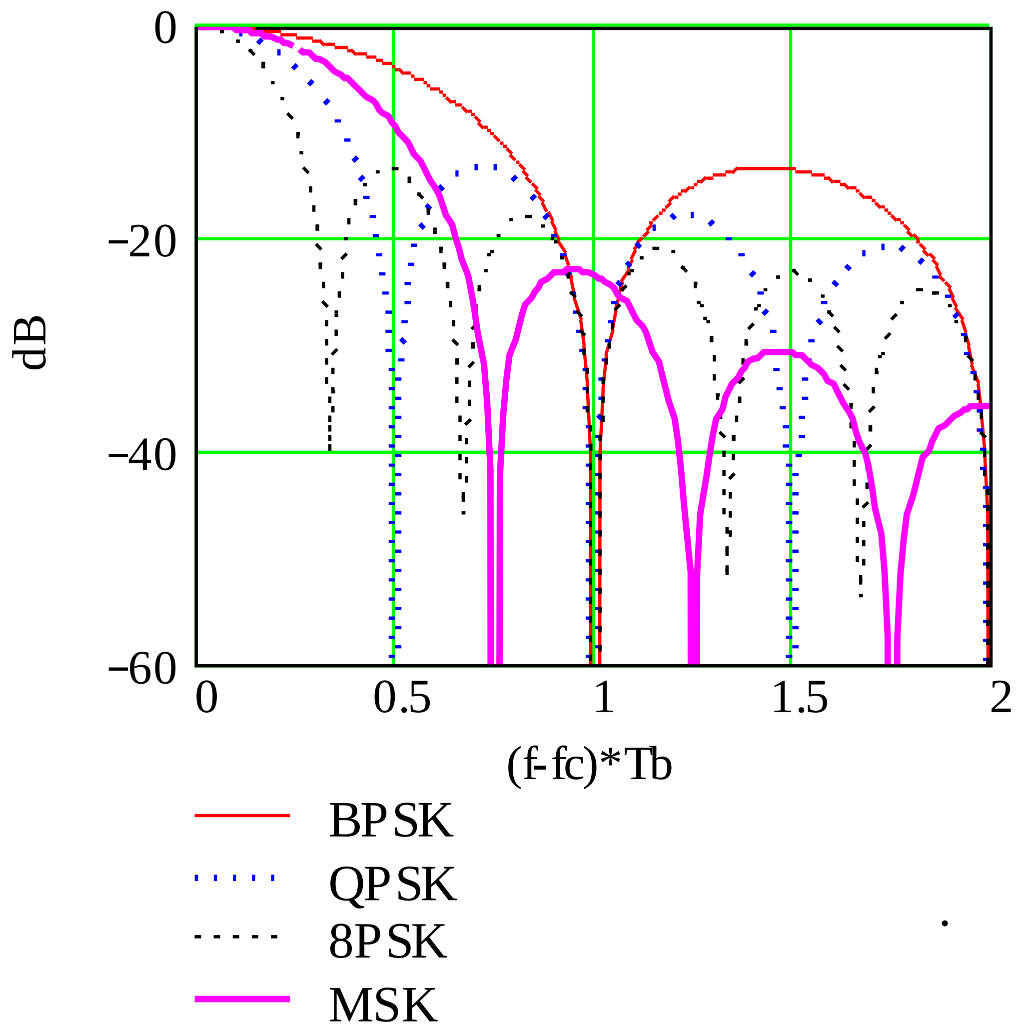


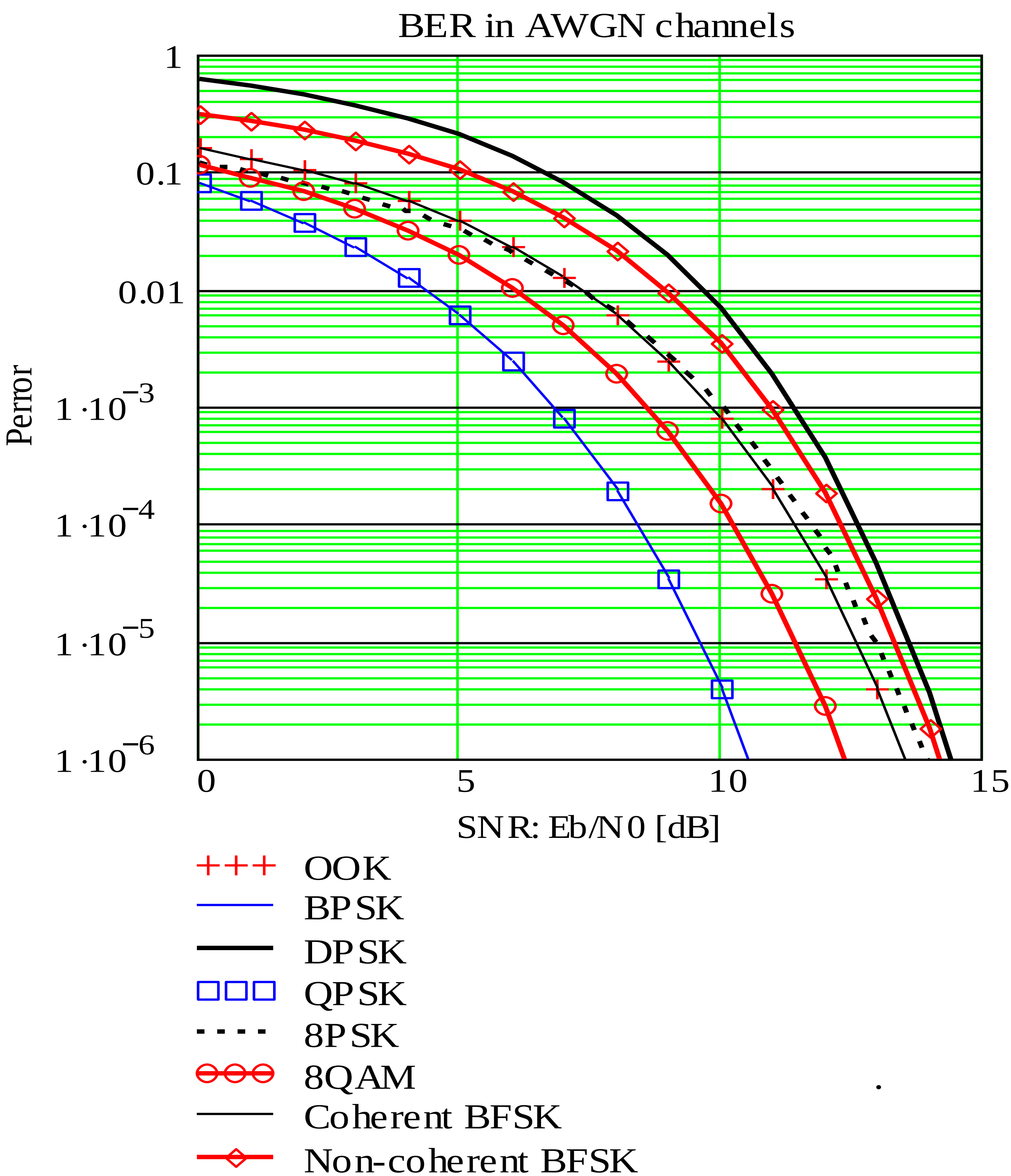
Figure 9.6 Bandwidth-efficiency plane.

Most high-rate systems use M-QAM , $M \geq 4$.

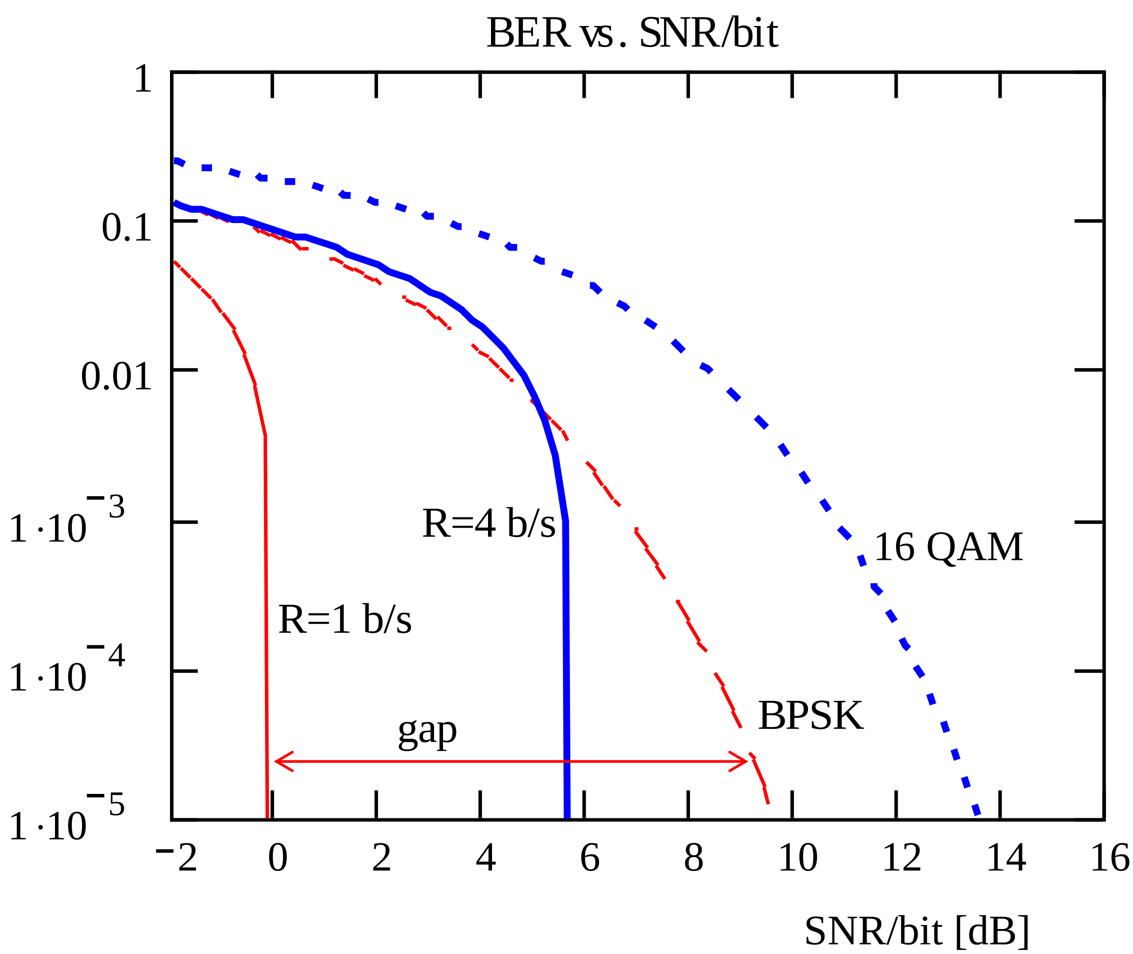
Spectrum of Digital Modulation (RF)



BER: Comparison of Various Modulation Formats



SNR Gap to Capacity



See e.g. D.J.C. MacKay, Information Theory, Inference and Learning Algorithms, Cambridge University Press, 2003, p.15, 162.

Maximum achievable rate with coding:

$$R_{\max} = \frac{\log_2(1+SNR)}{1-h(P_e)} \quad \text{[bit/ symbol]} \qquad (9.49)$$

$h(P_e) = -P_e \log_2 P_e - (1-P_e)\log_2(1-P_e)$ = binary entropy

SOME REFERENCES ON CHANNEL CODING/MODULATION:

Books:

E. Biglieri, Coding for Wireless Channels, Springer, 2005.

W.E. Ryan and S. Lin, *Channel Codes: Classical and Modern*. Cambridge, U.K.: Cambridge Univ. Press, 2009.

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State-of-the-Art (in Optical Communications)

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Record-High 17.3-bit/s/Hz Spectral Efficiency Transmission over 50 km Using Probabilistically Shaped PDM 4096-QAM

Samuel L.I. Olsson[1,*], Junho Cho[1], Sethumadhavan Chandrasekhar[1], Xi Chen[1], Ellsworth C. Burrows[1], and Peter J. Winzer[1]

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16384-QAM TRANSMISSION AT 10 GBD OVER 25-KM SSMF USING POLARIZATION-MULTIPLEXED PROBABILISTIC CONSTELLATION SHAPING

Xi Chen, Junho Cho, Andrew Adamiecki, and Peter Winzer

Nokia Bell Labs, Holmdel, New Jersey, United States

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10.66 Peta-Bit/s Transmission over a 38-Core-Three-Mode Fiber

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Summary

- Geometric representation of signals via signal space.
- Optimum receiver (MAP, ML) in the signal space.
- BPSK, QPSK, QAM.
- M-ary modulation formats. Comparisson.
- Power and bandwidth efficiency.
- BER and SER.
- Fundamental limits. Channel capacity.

Reading:

- Rappaport, Ch. 6 (expect 6.11, 6.12).
- Other books (see the reference list).

Note: Do not forget to do end-of-chapter problems. Remember the learning efficiency pyramid!

Appendix: Optimal Rx

Optimal Rx selects $m = m_k$ with largest a posteriori probability:

$$m = m_k \quad \text{if} \quad \Pr\{m_k|\mathbf{r}\} \geq \Pr\{m_i|\mathbf{r}\} \quad \forall i \neq k \qquad (9.50)$$

$$\Pr\{m_k|\mathbf{r}\} = \frac{\Pr\{\mathbf{r}|m_k\}\Pr\{m_k\}}{\Pr\{\mathbf{r}\}} \qquad (9.51)$$

To prove its optimality, observe that the probability of correct decision $\Pr\{c|m_k\}$ given that the Rx selects $m = m_k$ is $\Pr\{c|m_k\} = \Pr\{m_k|\mathbf{r}\}$, which is maximized by (9.50).

This rule also maximizes unconditional probability of correct decision $(P_c = 1 - P_e)$:

$$P_c = \int \Pr\{\mathbf{r}\}\Pr\{c|\mathbf{r}\}d\mathbf{r}, \quad \Pr\{c|\mathbf{r}\} = \sum_{k=1}^{M} \Pr\{m_k|\mathbf{r}\}\Pr\{m_k\} \qquad (9.52)$$

From (9.50), (9.51), the maximum a posteriori probability (MAP) decision rule follows:

$$\boxed{m = m_k \text{ if } \Pr\{\mathbf{r}|\mathbf{s}_k\}\Pr\{\mathbf{s}_k\} \geq \Pr\{\mathbf{r}|\mathbf{s}_i\}\Pr\{\mathbf{s}_i\} \ \forall i \neq k} \qquad (9.53)$$

since $\Pr\{\mathbf{s}_i\} = \Pr\{m_i\}$, $\Pr\{\mathbf{r}|\mathbf{s}_i\} = \Pr\{\mathbf{r}|m_i\}$, i.e. decide in favor of such m_k that maximizes a posteriori probability of observed $\mathbf{r}$.

From (9.15),

$$\Pr\{\mathbf{r}|\mathbf{s}_k\} = P_\xi(\mathbf{r}-\mathbf{s}_k) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{|\mathbf{r}-\mathbf{s}_k|^2}{N_0}\right) \qquad (9.54)$$

So that the decision rule in (9.53) becomes

$$m = m_k \text{ if } |\mathbf{r}-\mathbf{s}_k|^2 + c_k \le |\mathbf{r}-\mathbf{s}_i|^2 + c_i,\ \forall i \ne k \qquad (9.55)$$

where $c_k = -N_0 \ln \Pr\{m_k\}$.