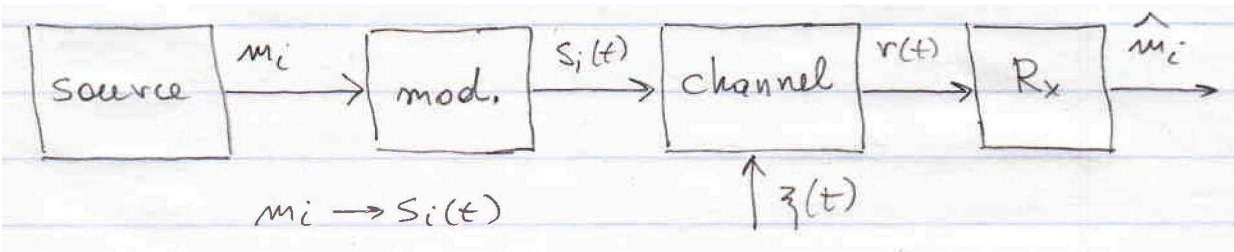
Optimal Receiver

System block diagram:



<u>System model under additive noise:</u>

$$r(t) = s_i(t) + \xi(t), \; 0 \le t \le T \tag{7.1}$$

The receiver:

$$\widehat{m_i} = D\{r(t)\} \tag{7.2}$$

Probability of error (for given m_i):

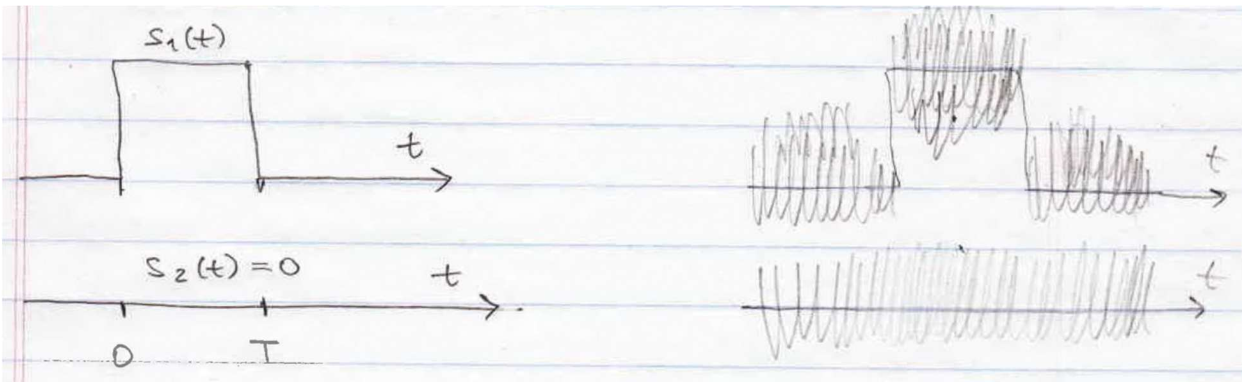
$$P_{ei} = \Pr\{\widehat{m_i} \neq m_i\} \tag{7.3}$$

Unconditional probability of error:

$$P_e = \Pr\{\widehat{m} \neq m\} = \sum_{i=1}^{N} P_{ei} \Pr\{m_i\} \overset{(a)}{=} \frac{1}{N} \sum_{i=1}^{N} P_{ei} \tag{7.4}$$

(a): if $\Pr\{m_i\} = \dfrac{1}{N} =$ equiprobable signaling.

- $P_e \rightarrow$ symbol error rate (very important).
- How many symbols are in error if n are transmitted? On average, $n_e = nP_e$.
- How to minimize $P_e \rightarrow$ Optimal Rx design.



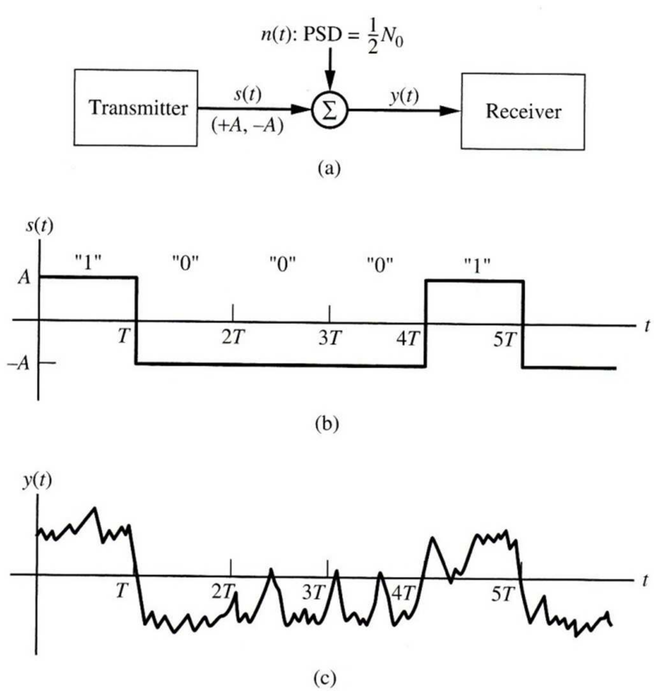


Figure 8.2
System model and waveforms for synchronous baseband digital data transmission.
(a) Baseband digital data communication system.
(b) Typical transmitted sequence. (c) Received sequence plus noise.

Ziemer, Tranter, Principles of Communications, Wiley, 2009.

- How to discriminate $s_1(t)$ and $s_2(t)$ under noise?
- Systematic optimal Rx design is based on probability theory (maximum likelihood etc.).

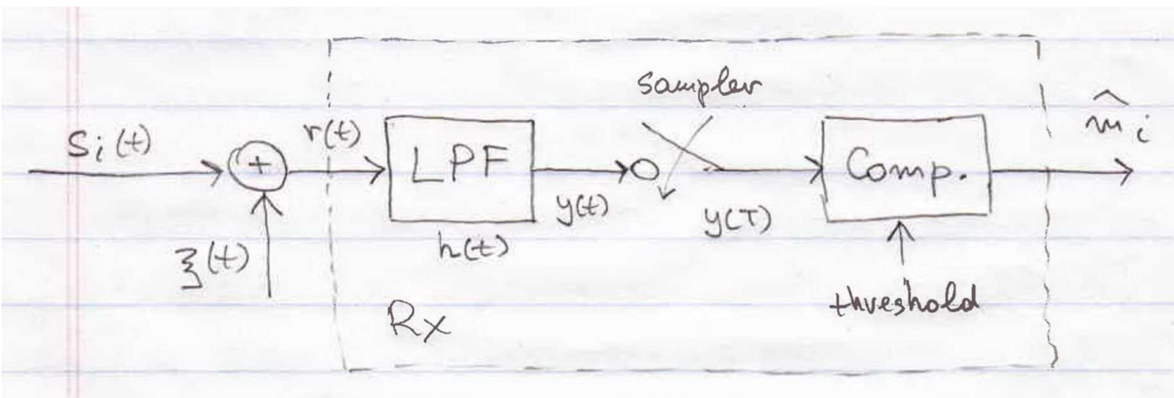
Baseband Binary Signaling (modulation)

Consider the simplest case first:

$$m_i = 1,0; \quad s_1(t) = p(t), \quad s_2(t) = -p(t) \qquad (7.5)$$

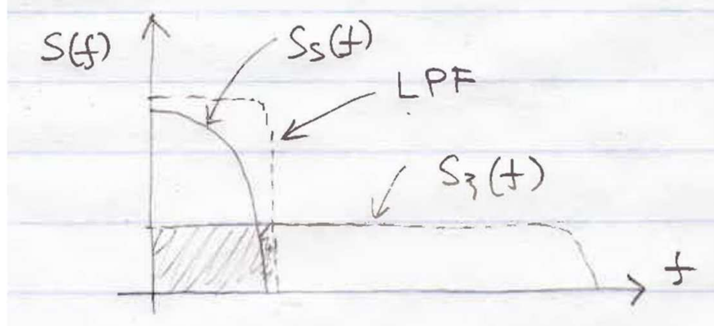
(baseband BPSK); $\quad s_i(t) = a_i p(t), \quad a_i = \pm 1.$

Optimal Rx structure (baseband):

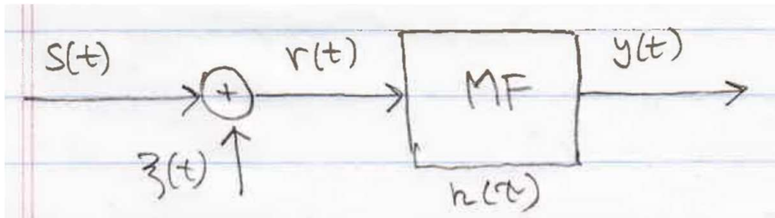


$$r(t) = s_i(t) + \xi(t); \quad y(t) = r(t) * h(t)$$
$$y(T) = y(t = T) \qquad (7.6)$$

* LPF = eliminate part of the noise
* Comparator + Sampler: make decision $(m_i = 1 \text{ or } m_i = 0)$.
* LPF: $P_e \downarrow$ as SNR $\uparrow$.



Matched Filter (MF)



MF: an optimal LPF maximizing the output SNR at sampling time *t*.

$$y(t) = h(t) * r(t) = \underbrace{\int_0^T s(\tau) h(t-\tau) d\tau}_{y_s(t)} + \underbrace{\int_{-\infty}^{\infty} \xi(\tau) h(t-\tau) d\tau}_{y_n(t)}$$
$$= y_s(t) + y_n(t) \tag{7.7}$$

<u>The sampled signal power at the output (at time *t*):</u>

$$P_s = |y_s(t)|^2 = \left| \int_0^T s(\tau) h(t-\tau) d\tau \right|^2 \tag{7.8}$$

<u>Noise power:</u>

$$P_n = \overline{|y_n(t)|^2} = \overline{y_n(t) y_n(t)^*}$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{\xi(\tau_1)\xi(\tau_2)} h(t-\tau_1) h(t-\tau_2) d\tau_1 d\tau_2 \tag{7.9}$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(\tau_1 - \tau_2) h(t - \tau_1) h(t - \tau_2) d\tau_1 d\tau_2$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} h^2(t - \tau) d\tau$$

where:

$\overline{\xi(\tau_1)\xi(\tau_2)} = R(\tau_1 - \tau_2) = \frac{N_0}{2}\delta(\tau_1 - \tau_2)$ = noise autocorrelation function;

$N_0 / 2 =$ noise power spectral density (2-sided);

$\overline{\xi(t)} = 0 =$ zero-mean.

This is Additive White Gaussian Noise (**AWGN**).

The output (baseband) SNR:

$$\Gamma = \frac{P_s}{P_n} = \frac{\left| \int_0^T s(\tau) h(t - \tau) d\tau \right|^2}{\frac{N_0}{2} \int_{-\infty}^{+\infty} h^2(t - \tau) d\tau} \tag{7.10}$$

Matched filter $\rightarrow$ maximize the output SNR:

$$\max_{h(\tau)} \Gamma \tag{7.11}$$

How? → <u>Cauchy – Schwartz inequality:</u>

$$\left| \int_{-\infty}^{+\infty} x(t) y(t) dt \right|^2 \le \int_{-\infty}^{+\infty} x^2(t) dt \int_{-\infty}^{+\infty} y^2(t) dt \qquad (7.12)$$

with = if $x(t) = \alpha y(t)$, α= constant (any), so that

$$\Gamma = \frac{\left| \int_{-\infty}^{\infty} s(\tau) h(t-\tau) d\tau \right|^2}{\frac{N_0}{2} \int_{-\infty}^{+\infty} h^2(t-\tau) d\tau} \overset{(a)}{\le} \frac{2}{N_0} \int_0^T s^2(\tau) d\tau = \frac{2E_s}{N_0} \qquad (7.13)$$

where $E_s = \int_0^T s^2(\tau) d\tau$ = signal energy.

Inequality (a) becomes the equality if $s(\tau) = \alpha h(t-\tau)$, i.e. if

$$\boxed{h(\tau) = \alpha s(t-\tau)} \qquad (7.14)$$

where α = arbitrary constant (does not affect the SNR), and

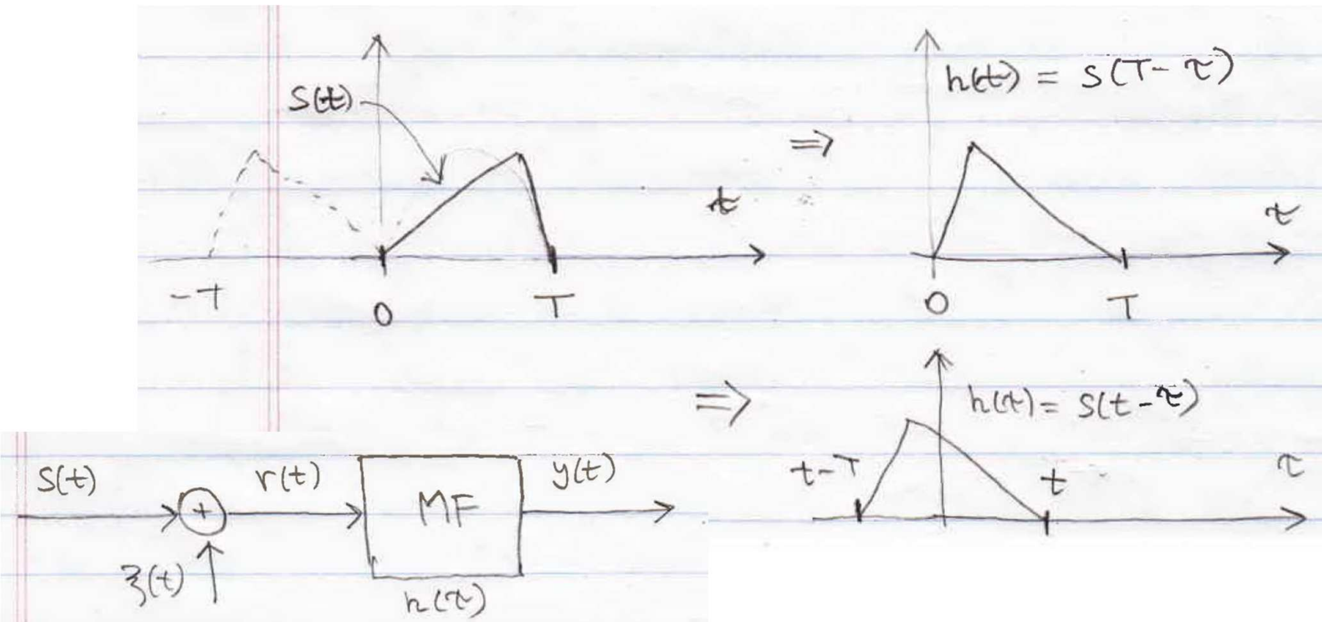
$$\boxed{\Gamma = \frac{2E_s}{N_0} = \Gamma_{max}} \qquad (7.15)$$

(7.14) = the impulse response of the MF.
(7.15) = the maximum output SNR achieved by this filter.

Best Sampling Time

$t =$ sampling time. To make a causal filter, set $t = T$:

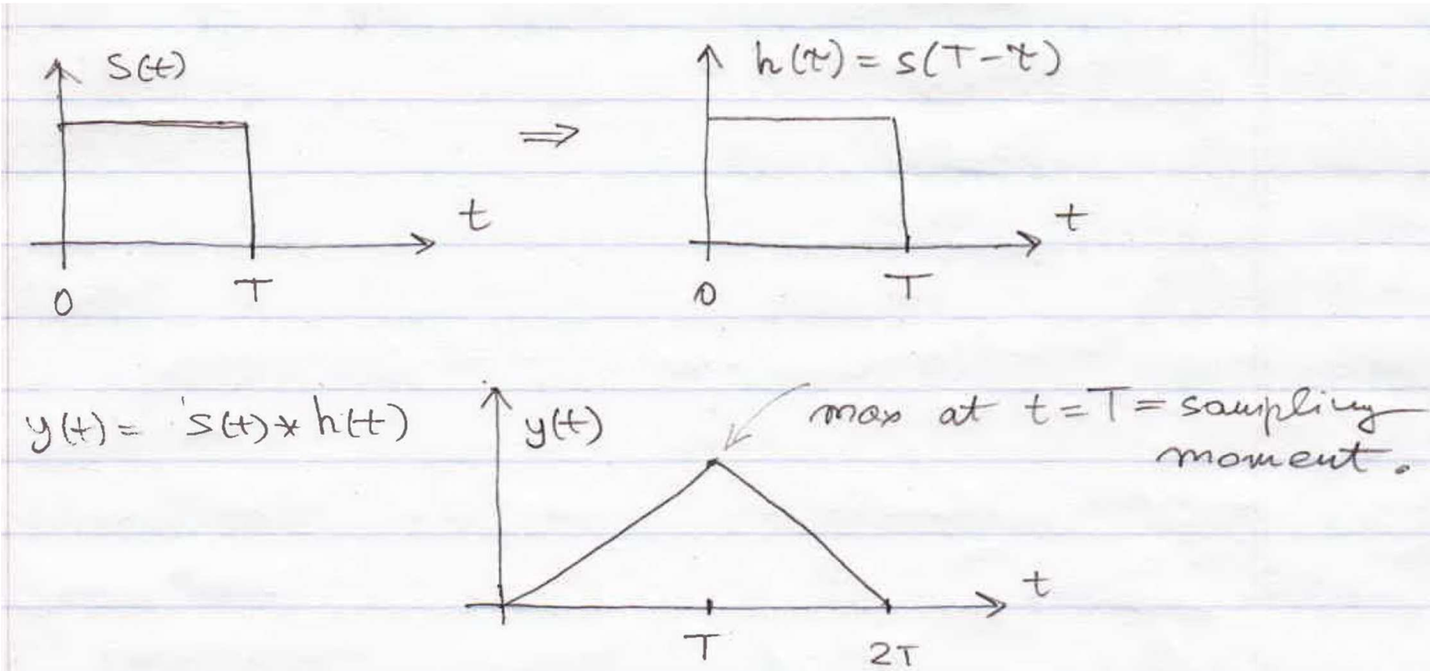
$$\boxed{h(\tau) = \alpha s(T - \tau)} \tag{7.16}$$

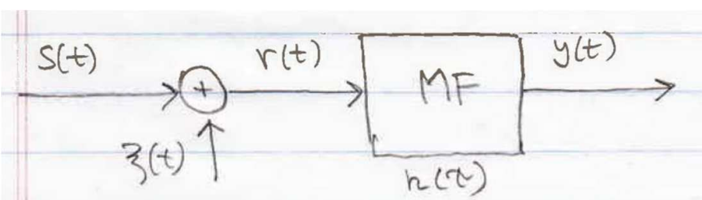


$$y(t) = h(t) * r(t) = \alpha \int_{t-T}^{t} r(\tau) s(T - t + \tau) d\tau \tag{7.17}$$

Matched Filter: An Example

Rectangular pulse:





$$y(t) = h(t) * r(t) = \alpha A \int_{t-T}^{t} r(\tau)\, d\tau, 0 \le t \le T, \quad (\alpha = \frac{1}{AT}).$$

$$y(T) = h(t) * r(t)\Big|_{t=T} = \alpha A \int_{0}^{T} r(\tau)\, d\tau$$

$$= \alpha A^2 T = \alpha E_s$$

(no noise).

Properties of the MF

- max $y(t)$ at the sampling instant $t = T$;
- pulse shape is not preserved;
- output SNR depends on the signal energy, but not on the pulse shape:

$$\Gamma = \Gamma_{\max} = \frac{2E_s}{N_0}; \quad E_s = \int_0^T s^2(\tau)\,d\tau,$$

i.e. many pulses of different shapes but same energy will provide the same SNR at the output of their MFs.

Matched Filter in Frequency Domain

Impulse response of the MF:

$$h(\tau) = \alpha s(T - \tau) \tag{7.18}$$

Frequency response:

$$H(f) = FT\{h(\tau)\} = \alpha S^*(f) e^{-j\omega T} \tag{7.19}$$

$\omega = 2\pi f$, i.e. same magnitude response (if $\alpha = 1$):

$$|H(f)| = |S(f)| \tag{7.20}$$

and phase response:

$$\begin{aligned} &\varphi_H(f) = -\left(\varphi_S(f) + 2\pi f T\right) \\ \Rightarrow\ &\varphi_H(f) + \varphi_S(f) + 2\pi f T = 0 \end{aligned} \tag{7.21}$$

Interpretation of (7.21)?

$$S_y(f) = H(f)S(f) = |S(f)|^2 e^{-j\omega T} \tag{7.22}$$

$$y(t = T) = \int_{-\infty}^{+\infty} |S(f)|^2 \, df \tag{7.23}$$

$\rightarrow$ coherent combining of all frequencies at $t = T$.

Probability of Error Analysis
(baseband BPSK)

After matched filtering and sampling:

$$y = y_s + y_n \tag{7.24}$$

where: $y_s = a_i \alpha \int_0^T s^2(\tau) d\tau = a_i \alpha E_s$ = signal part.

$a_i = \pm 1$ = message.

$$y_n = \alpha \int_0^T \xi(\tau) h(T-\tau) d\tau = \alpha \int_0^T \xi(\tau) s(\tau) d\tau$$

= noise (at the output).

The MF impulse response $h(\tau) = \alpha s(T-\tau)$;
α = normalization constant (does not affect the performance);
choose $\alpha = 1/\sqrt{E_s}$ (unit energy gain of the MF), so that:

$$y_s = a_i \sqrt{E_s} \tag{7.25}$$

y_n is Gaussian (why?) with mean and variance as follows:

$$\overline{y_n} = \alpha \int_0^T \overline{\xi(\tau)} s(\tau) d\tau = 0$$

$$\operatorname{var}(y_n) = \overline{y_n^2} = \frac{N_0}{2} \alpha^2 \int_0^T s^2(\tau) d\tau = \frac{N_0}{2} = \sigma_0^2 \tag{7.26}$$

i.e. $y_n \sim N\left(0, \frac{N_0}{2}\right)$, $y \sim N\left(\sqrt{E_s}, \frac{N_0}{2}\right)$.

Probability of error:

$$P_e = \Pr\left\{\widehat{m} \neq m\right\} = P_{e|1}\Pr\{m_i = 1\} + P_{e|0}\Pr\{m_i = 0\} \qquad (7.27)$$

$$m_i = 1 \;\rightarrow\; a = 1 \;\rightarrow\; y_s = \sqrt{E_s}$$
$$m_i = 0 \;\rightarrow\; a = -1 \;\rightarrow\; y_s = -\sqrt{E_s}$$

If $m_i = 1$, an error occurs if $\widehat{m_i} = 0$, i.e. if $y < y_{th}$, so that

$$P_{e|1} = \Pr\left\{y < y_{th}\,|_{m_i=1}\right\} = \Pr\left\{\frac{y_n}{\sigma_0} < \frac{y_{th} - \sqrt{E_s}}{\sigma_0}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\tilde{y}_{th}} e^{-x^2/2}dx \qquad (7.28)$$

$$= Q\left(\frac{\sqrt{E_s} - y_{th}}{\sigma_0}\right)$$

where

$$\boxed{Q(x) = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-t^2/2}dt} \qquad (7.29)$$

is Q-function (CCDF of $z \sim N(0,1)$; $\text{CCDF} = 1 - \text{CDF}$; and

$$\tilde{y}_{th} = \left(y_{th} - \sqrt{E_s}\right)/\sigma_0$$

is the normalized threshold,

$$P_{e|1} = \text{prob. of error if } m_i = 1.$$
$$P_{e|0} = \text{prob. of error if } m_i = 0.$$

Similar reasoning results in

$$P_{e|0} == Q\left(\frac{\sqrt{E_s} + y_{th}}{\sigma_0}\right) \tag{7.30}$$

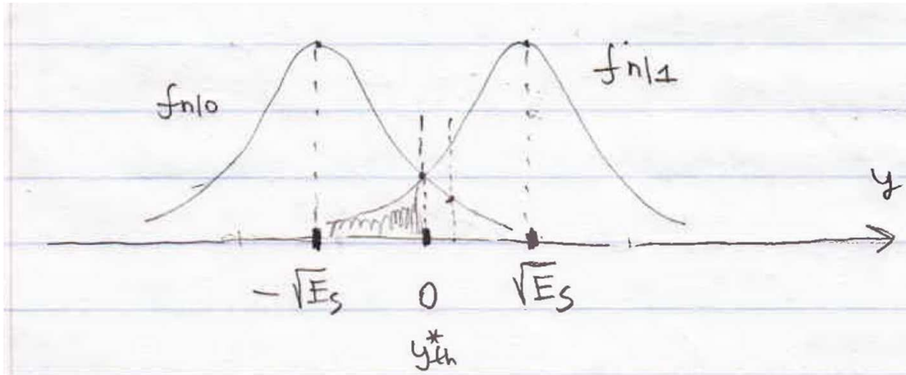
Often the source is such that $\Pr\{m_i = 1\} = \Pr\{m_i = 0\} = 1/2$ (why?), so that

$$P_e = \frac{1}{2}\left(P_{e|1} + P_{e|0}\right) = \frac{1}{2}\left\{Q\left(\frac{\sqrt{E_s} - y_{th}}{\sigma_0}\right) + Q\left(\frac{\sqrt{E_s} + y_{th}}{\sigma_0}\right)\right\} \tag{7.31}$$

Optimal threshold: Select y_{th} such that $P_e \to \min$:

$$\min_{y_{th}} P_e\left(y_{th}\right) = Q\left(\frac{\sqrt{E_s}}{\sigma_0}\right), \quad y_{th}^* = 0 \tag{7.32}$$

Q: prove (7.32)!



Finally,

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = Q\left(\sqrt{2\gamma}\right)} \tag{7.33}$$

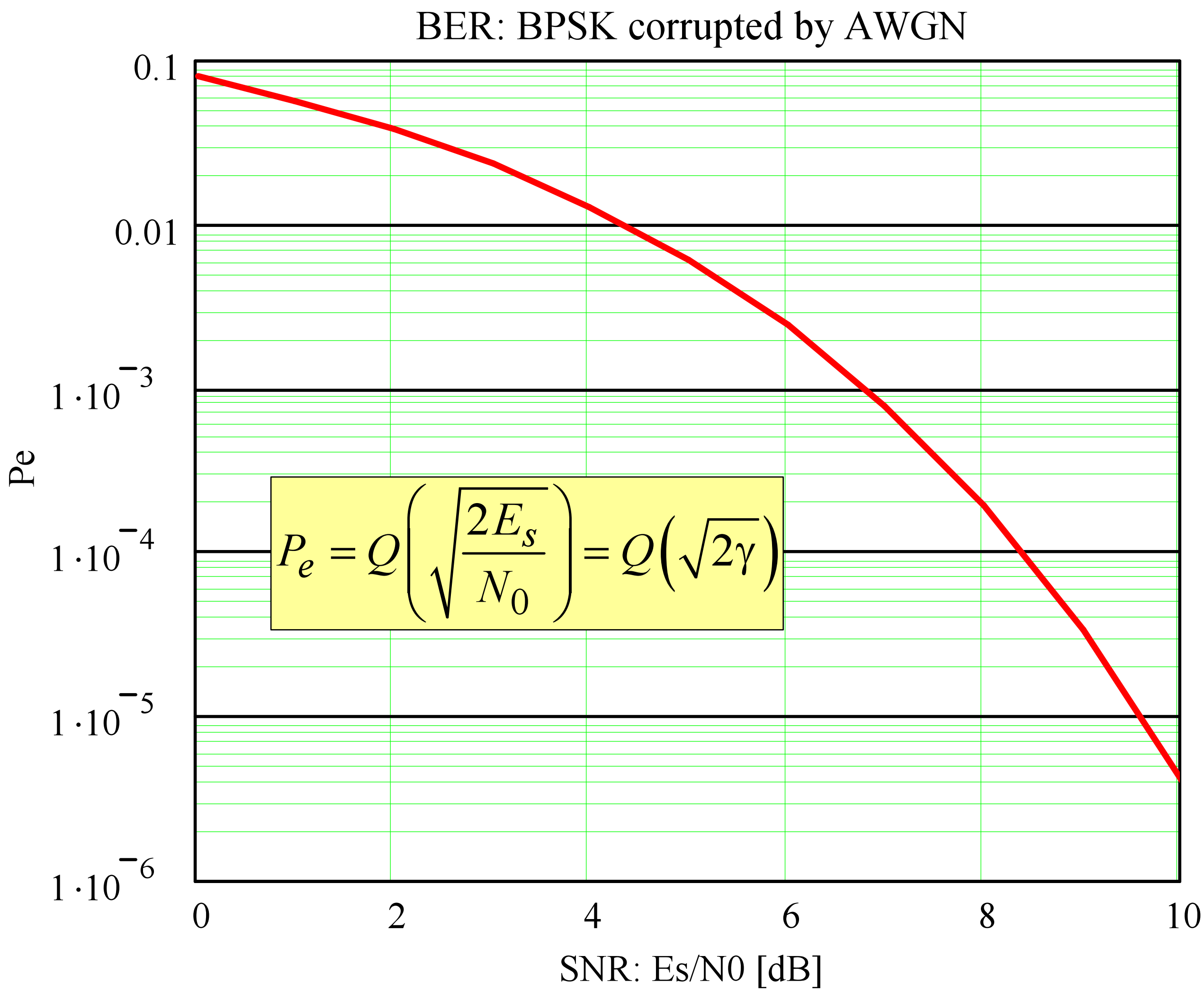
where $\gamma = \dfrac{E_s}{N_0}$ is the RF (bandpass) output SNR:

$$\gamma = \frac{E_s}{N_0} = \frac{E_s \Delta f}{N_0 \Delta f} = \frac{E_s / T}{N_0 \Delta f} = \frac{P_s}{P_{n,RF}} \tag{7.34}$$

Recall that: $\Delta f_{RF} = 2\Delta f_{BB} = 2\Delta f$.

Q.: What is the input SNR?

Probability of Error for Baseband BPSK



$$P_e = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = Q\left(\sqrt{2\gamma}\right)$$

Properties:

- P_e depends on E_s, not on the signal shape. (all signals of same $E_s \rightarrow$ same P_e)
- P_e depends on the SNR = E_s / N_0, not on E_s or N_0 individually (i.e. weak signal is OK in weak noise).
- $P_e \downarrow \text{SNR}$.
- $P_e(\gamma = 0) = \frac{1}{2} \rightarrow$ explain! (why not 1?).
- ISI design: signal shape is important, not energy
  P_e design: energy is important, not shape.

***

Q.: Do the same analysis for OOK, i.e.

$$m = 1 \rightarrow a = 1$$

$$m = 0 \rightarrow a = 0$$

find P_e, y_{th}^*.

Properties of Q-function:

This function is very important in digital communications.

Useful relation to the erfc function:

$$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right), \; erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \qquad (7.35)$$

Asymptotically,

$$\begin{aligned} erfc(x) &\approx \frac{1}{\sqrt{\pi} x} e^{-x^2}, \; x \to \infty \\ Q(x) &\approx \frac{1}{\sqrt{2\pi} x} e^{-x^2/2} \end{aligned} \qquad (7.36)$$

Properties and bounds:

$$Q(x) \le \frac{1}{\sqrt{2\pi} x} e^{-x^2/2}, \; x \ge 0 \qquad (7.37)$$

$$Q(x) \le \frac{1}{2} e^{-x^2/2}, \; x \ge 0 \qquad (7.38)$$

$$Q(0) = \frac{1}{2} \qquad (7.39)$$

$$Q(-x) = 1 - Q(x), \; x \ge 0 \qquad (7.40)$$

Q.: $Q(\infty) = ? \quad Q(-\infty) = ?$

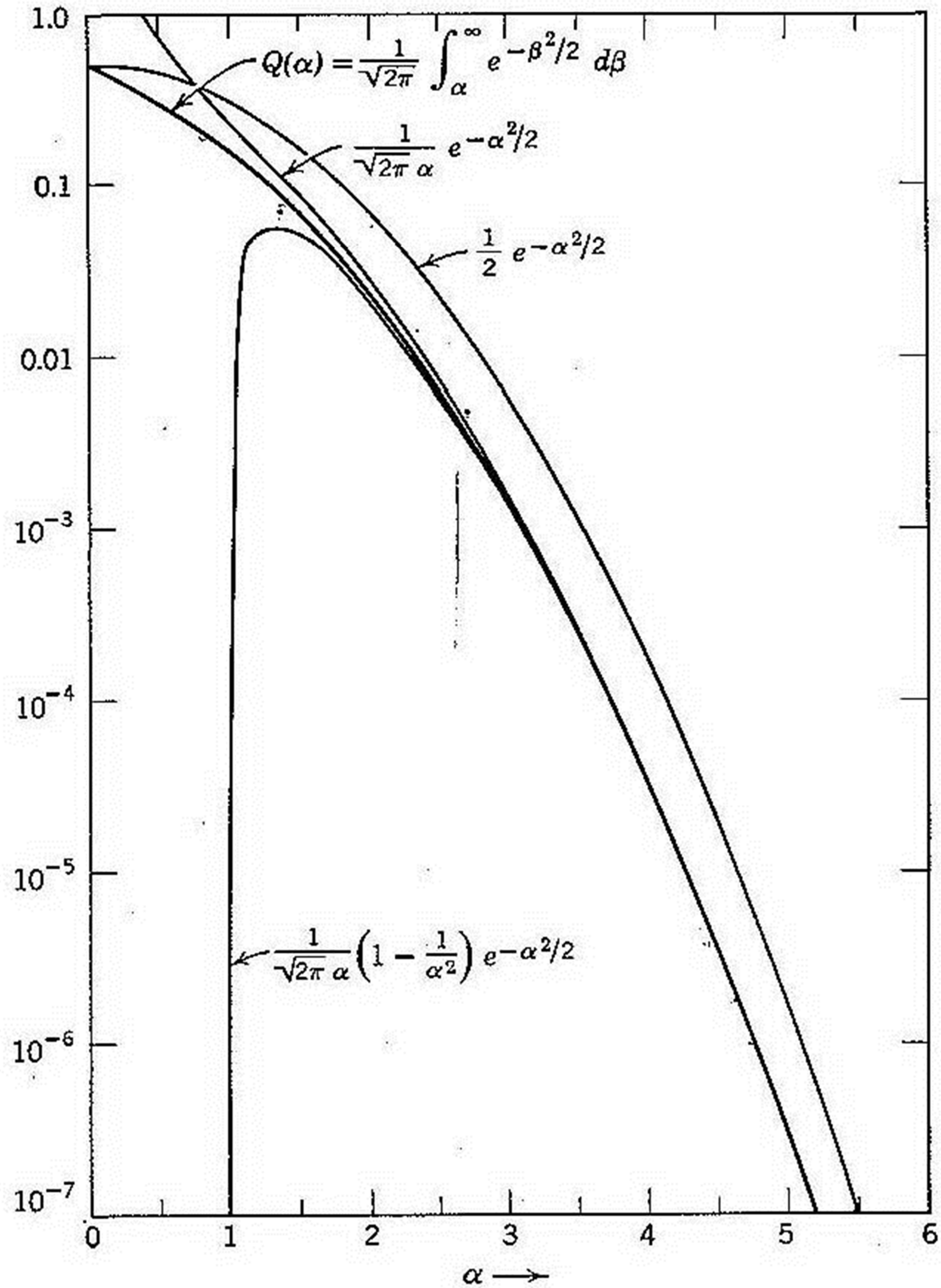
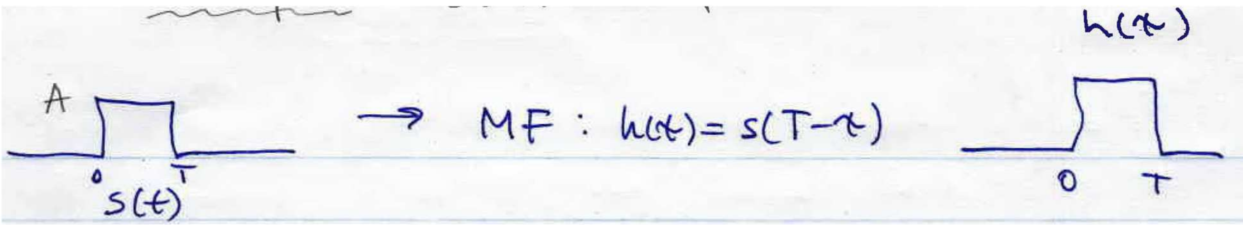
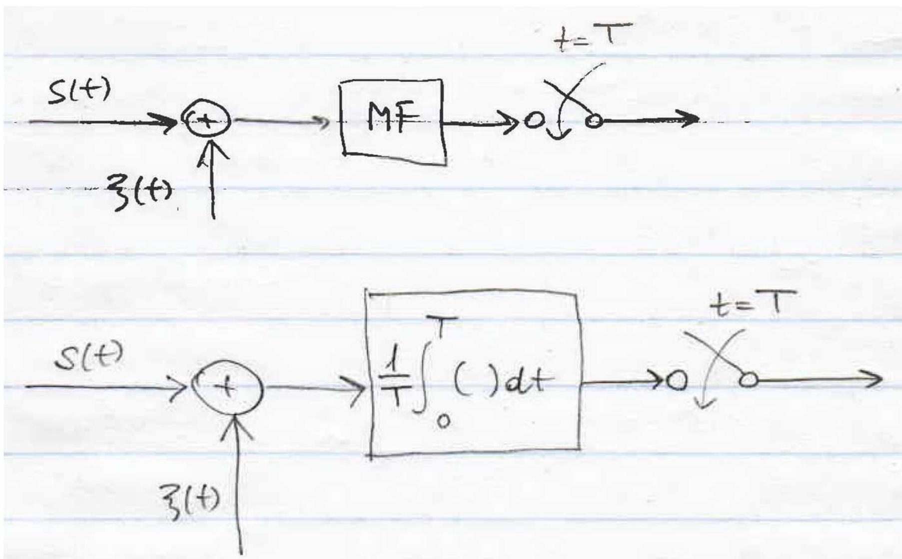


Figure 2.36   The function $Q(\alpha)$ and three bounds.

J.M. Wozencraft, I.M. Jacobs, Principles of Communication Engineering, Wiley: New York.

Example: optimum Rx for a rectangular pulse





$$s(t) = A \cdot \Pi(t/T),\ h(t) = \alpha \cdot s(T-t) = \frac{1}{T}\Pi(t/T) \quad (7.41)$$

Signal sample: $$y_s = \frac{1}{T}\int_0^T s(t)\,dt = A; \quad (7.42)$$

Noise sample: $$y_n = \frac{1}{T}\int_0^T \xi(t)\,dt \to \overline{y_n} = 0; \quad (7.43)$$

$$\mathrm{var}(y_n) = \overline{y_n^2} = \frac{1}{T^2}\int_0^T\int_0^T \overline{\xi(t_1)\xi(t_2)}\,dt_1 dt_2$$

$$= \frac{1}{T^2}\int_0^T\int_0^T \sigma_0^2 \delta(t_1 - t_2)\,dt_1 dt_2 \tag{7.44}$$

$$= \frac{1}{T^2}\int_0^T \sigma_0^2 dt_1 = \frac{\sigma_0^2}{T}$$

$$\sigma_n = \frac{\sigma_0}{\sqrt{T}}, E_s = A^2 T \tag{7.45}$$

$$P_e = Q\left(\sqrt{\frac{A^2T}{\sigma_0^2}}\right) = Q\left(\sqrt{2\gamma}\right),\ \gamma = \frac{A^2T}{N_0} = \frac{E_s}{N_0} \tag{7.46}$$

where $\sigma_n^2 = \sigma_0^2 / T$ = noise sample variance, $\sigma_0^2 = N_0 / 2$.

Summary

- Optimal reception in noise.
- Optimal Rx structure (binary signaling).
- Matched filter. Its properties.
- Probability of error analysis.

Reading:

- Rappaport, Ch. 6 (6.1-6.10).
- L.W. Couch II, Digital and Analog Communication Systems, 7th Edition, Prentice Hall, 2007. (other editions are OK as well)
- Other books (see the reference list).

Note: Do not forget to do end-of-chapter problems. Remember the learning efficiency pyramid!