ELG3175: Introduction to Communication Systems

Introduction to Information Theory

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Big Picture

- Information theory: quantifies info. generation, transmission, storage
- Determines fundamental limits to reliable communication
- Two fundamental parts:
  - **Source coding**: compression
  - **Channel coding**: reliable communication
- Key quantities:
  - entropy
  - mutual information
  - channel capacity

A Simple Guessing Game

Suppose someone chooses one message and you try to guess it.

- If there are only 2 equally likely messages, one yes/no question is enough
- If there are 4 (or 8) equally likely messages, you need more questions
- If one possibility is much more likely than the others, guessing becomes easier

Key lesson:

$$\textbf{more uncertainty} \implies \textbf{more information needed to describe the outcome}$$

What is Information?

- A highly predictable event carries little information
- A surprising event carries more information
- Information can be viewed as **reduction in uncertainty**

Examples:

- fair coin flip $\rightarrow$ uncertain
- deterministic output $\rightarrow$ no surprise
- rare event $\rightarrow$ more informative

Self-Information

For an event $X = x$ with probability $p(x)$, define

$$I(x) = \log \frac{1}{p(x)} = -\log p(x).$$

- rare events have larger self-information
- if $p(x) = 1$, then $I(x) = 0$
- independent events add:

$$I(x, y) = I(x) + I(y)$$

- base-2 logarithm gives units of **bits**

Entropy

- entropy is the **average self-information**

$$H(X) = -\sum_{m=1}^{M} p(a_m) \log p(a_m),\ \ X \in\ [a_1, a_2, ...a_M]$$

- it measures the uncertainty in X
- it is also the fundamental limit of data compression
- depends on probabilities $p(a_m) = p_m$ only, not on a_m:

$$H(X) = -\sum_{m} p_m \log p_m$$

Properties of Entropy I

- **Nonnegativity:**
$$H(X) \geq 0$$
- **Deterministic case:**
$$H(X) = 0 \iff X \text{ is deterministic}$$
- **Upper bound:** if $X \in [a_1, a_2, ...a_M]$, then
$$H(X) \leq \log M$$
- Equality holds when X is uniform on $[a_1, a_2, ...a_M]$

An Example: A Binary Source

Let $X \sim \mathrm{Bernoulli}(p)$, so

$$P(X = a_1) = p, \qquad P(X = a_2) = 1 - p.$$

$$H(X) = H_2(p) = -p \log_2 p - (1 - p) \log_2(1 - p).$$

If $p = 0.5$, then $H(X) = 1$ bit.
If $p = 0.9$, then

$$H(X) = -0.9 \log_2 0.9 - 0.1 \log_2 0.1 \approx 0.47 \text{ bits}.$$

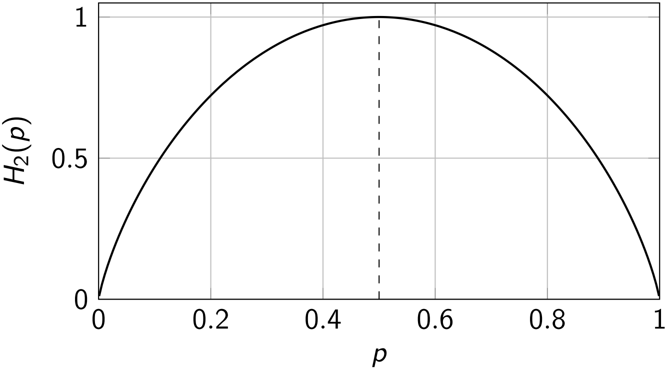
Interpretation: a heavily biased source is easier to guess, so it has lower entropy.

Q: what is the best guessing strategy? Probability of error?

Binary Entropy Function

For $X \sim \mathrm{Bernoulli}(p)$,

$$H(X) = H_2(p) = -p \log p - (1-p)\log(1-p).$$



- maximum at $p = 0.5$, symmetric about $p = 0.5$

Joint and Conditional Entropy

- $H(X, Y)$: uncertainty in the pair (X, Y)

$$H(X, Y) = -\sum_{x,y} p(x, y) \log p(x, y)$$

- $H(X|Y)$: uncertainty in X after observing Y

$$H(X|Y) = \sum_{y} p(y) \, H(X|Y = y).$$

- **chain rule**:

$$H(X, Y) = H(Y) + H(X|Y)$$

- **Q:** prove it

Properties of Entropy II

- **Chain rule:**

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

- **Conditioning reduces entropy:**

$$H(X|Y) \leq H(X)$$

- **Independence implies additivity:** if X and Y are independent, then

$$H(X,Y) = H(X) + H(Y)$$

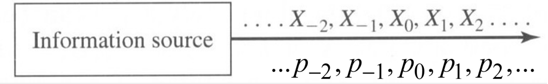
- **Q:** prove all of the above

Data Compression (Source Coding)

- the most important (and useful) application of entropy
- how to compress file/data stream losslessly?
- what is the fundamental limit of compression?
- simple yet fundamental model: **discrete memoryless source** (DMS):

$$X^n = [X_1, X_2, .., X_n],\ X_i \sim \text{i.i.d.} \tag{1}$$
$$X_i \in \{a_1, a_2, .., a_M\},\ \Pr\{X_i = a_m\} = p_m$$

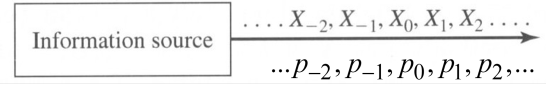
where $\{a_1, a_2, .., a_M\}$ is the source alphabet



- e.g. bandlimited analog (continuous) source ⇒ sampling theorem ⇒ discrete-time source

Data Compression (Source Coding)

- the most important (and useful) application of entropy



- **Q1**: what is the minimum data rate needed to transmit this source losslessly?
- **Q2:** how many bits are needed to save X^n in a file?

Data Compression (Source Coding)

- the most important (and useful) application of entropy
- **Source Coding Theorem**[1]: any DMS can be transmitted losslessly with rate R [bit/sym.] if $R > H(X)$. Lossless transmission is not possible if $R < H(X)$.
- equivalently, for large n, sequence X^n can compressed losslessly to any N_b [bits] if $N_b > nH(X)$. Lossless compression is not possible if $N_b < nH(X)$.
- i.e. $H(X)$ is a **sharp boundary** of what is possible and what is not in lossless data compression
- $H(X) \approx$ **absolutely minimum number of bits/symbol** required to transmit/save the source losslessly

[1] this is a "rough" formulation that gives a big picture; consult any IT book for precise details and definitions of source codes.

Data Compression (Source Coding)

- **Source Coding Theorem**[2]: any DMS can be transmitted/stored losslessly with rate R if $R > H(X)$. Lossless transmission/storage is not possible if $R < H(X)$.

- proof: via LLN
- can be extended to sources with memory (less bits are needed)
- can be applied to continuous (analog) sources using quantization
- used extensively in practice (.zip, .jpeg, .mp3, .mp4, etc.)

[2] T.M. Cover, J.A. Thomas, Elements of Information Theory, Wiley, 2006.

An Example: A Binary Source

Consider DMS with $X_i \sim \mathrm{Bernoulli}(p)$ and $p = 0.9$,

$$P(X = a_1) = p, \qquad P(X = a_2) = 1 - p.$$

and $n = 10^6$.

- **Q1:** minimum R to transmit the source losslessly?
- **Q2:** minimum number of bits to store it losslessly?
- **Q3:** how do these change if (a) $p = 0.5$? (b) $p = 1$? (c) $p = 0$?
- **Q4:** how do the answers to Q1 and Q2 change if $X_1 = X_2 = ... = X_n$?

Example: Rate of a Bandlimited Source

- Baseband, bandlimited source, $F_{\max} = 4$ kHz, sampled at Nyquist rate
- Samples are quantized to $[-2, -1, 0, 1, 2]$, and the corresponding probabilities are $[1/16,\ 1/8,\ 1/2,\ 1/4,\ 1/16]$.
- What is the minimum rate [bit/s] needed to transmit the source ?

$$H(X) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{2}{16}\log 16 = \frac{15}{8} \text{ bit/sample}$$

$$R = H(X) \cdot f_s = H(X) \cdot 2F_{\max} = 15 \text{ kbit/s}$$

- **Q1:** how the answer would change for a uniformly-distributed source?
- **Q2:** if only 2 symbols have non-zero probability? only one?

Summary

- Introduction to information theory
- Entropy: measure of information (and uncertainty)
- Properties
- Joint/conditional entropies
- Data compression (source coding). Fundamental limit
- Discrete memoryless source.

Reading

- R.E. Ziemer, W.H. Tranter, Principles of Communications, Wiley, New York, 2009. Ch. 11.
- B.P.Lathi, Z. Ding, Modern Digital and Analog Communication Systems, Oxford University Press, 2009.
- T.M. Cover, J.A. Thomas, Elements of Information Theory, Wiley, 2006.

Note: Do <u>not</u> forget to do end-of-chapter problems. Remember the learning efficiency pyramid!