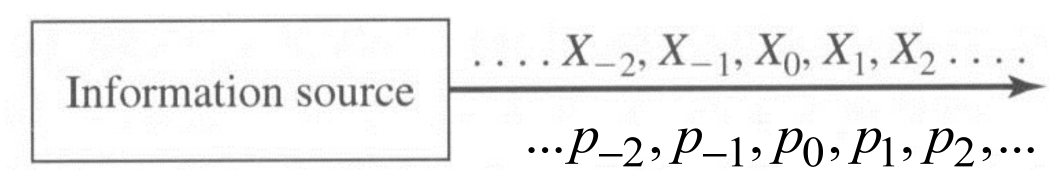
Introduction to Information Theory

- All communication systems are designed to transmit information.
- What is information? Qualitative (intuitive) description (knowledge about something) is not enough -> quantitative definition is required.
- Start with an information source, which produces outputs that are not known to the receiver in advance.
- Bandlimited analog (continuous) source -> sampling theorem -> discrete-time source.
- Simple model: discrete memoryless source (DMS):



Quantitative Measure of Information

- Defined in such a way that certain intuitive properties are satisfied.
- There is a probability associated with each source output: $[x_1 \ldots x_n]$ -> $[p_1 \ldots p_n]$. Which output conveys more information, highly probable or less probable?
- Information measure should be decreasing function of probability -> *Least probable output conveys most information*. It should also be a smooth function.

$$\boxed{I_i = I(p_i), \;\; p_1 < p_2 \leftrightarrow I_1 > I_2}$$

- Total information measure of two independent events is the sum of individual information measures:

$$\boxed{p = p_1 p_2 \Rightarrow I(p) = I(p_1) + I(p_2)}$$

Information Measure

- The only function that satisfies the properties above is the logarithm: $\boxed{I(x_i) = -\log(p(x_i)) \Rightarrow I = -\log_2 p \text{ [bits]}}$
- The base of logarithm is not important. If the base is 2, information is measured in bits.
- Important properties:

$$I(x_i) = 0 \text{ if } p(x_i) = 1$$
$$I(x_i) \geq 0$$
$$I(x_i) > I(x_j) \text{ if } p(x_i) < p(x_j)$$
$$I(x_i x_j) = I(x_i) + I(x_j) \text{ if } x_i \text{ \& } x_j \text{ are independent}$$

- Q.: What happens if p_i=0 ?

Average Information & Entropy

- Communication system: long sequences are transmitted.
- Entropy – average information content of the source per symbol:

$$H(X) = E_x[I(X)] = E_x[-\log p(X)] = -\sum_{i=1}^{N} p_i \log p_i$$

- It is a measure of uncertainty about x (on average): the more is known about x, the less is the entropy
- Example: uniform random variable,

$$x_i, i = 1, \ldots, n \rightarrow p_i = 1/n \rightarrow H(X) = ?$$

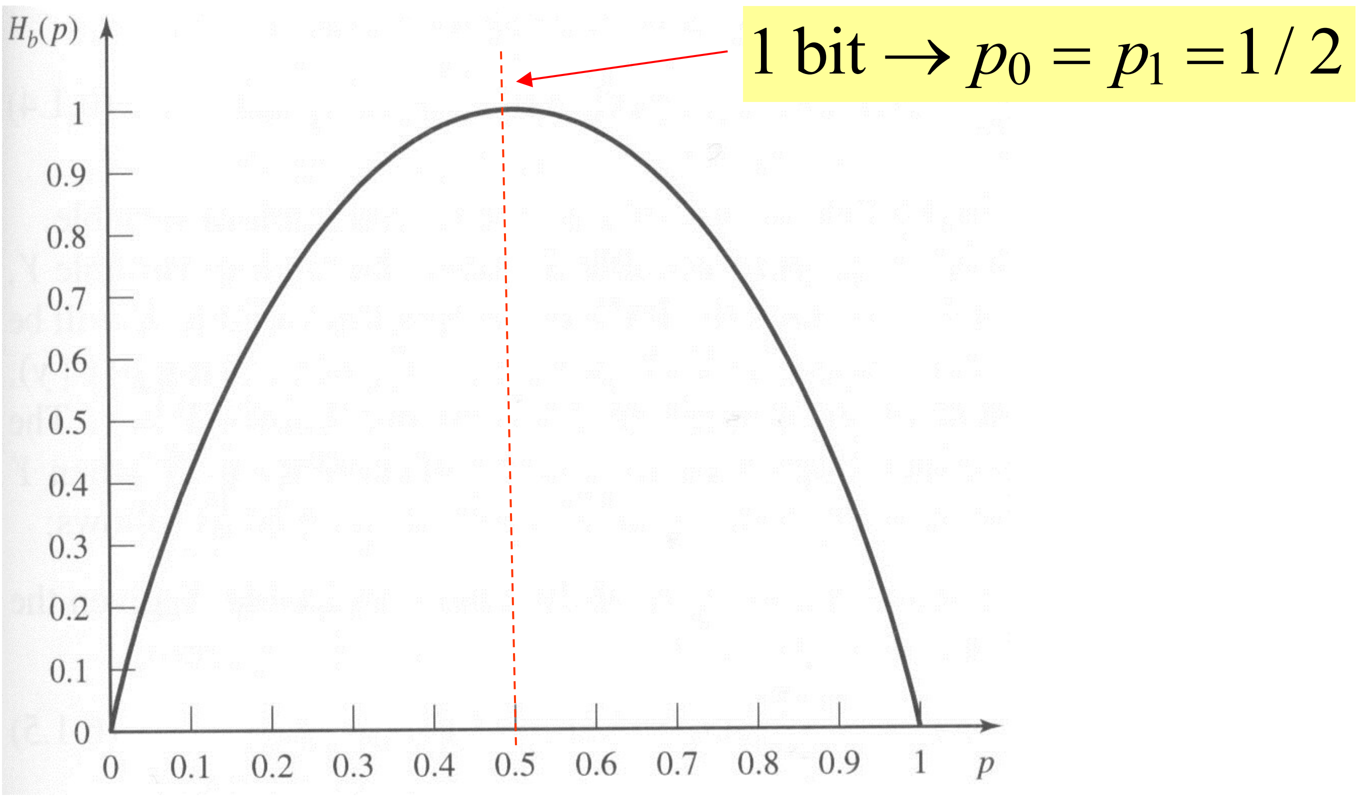
- Note that $0 \le H(\mathrm{X}) \le \log n$

When lower bound is achieved? Upper?

Example: Binary Memoryless Source

- Two possible outcomes, x_1 & x_2 -> p_1=p & p_2=1-p :

$$H(\mathrm{X}) = -p\log p - (1-p)\log(1-p)$$



J.Proakis, M.Salehi, Communications Systems Engineering, Prentice Hall, 2002

Example: Rate of a Bandlimited Source

- Consider a baseband, bandlimited source, F_{max}=4 kHz, sampled at Nyquist rate.
- Assume that the samples are quantized to [-2,-1,0, 1, 2], and the corresponding probabilities are [1/16, 1/8, 1/2, 1/4, 1/16].
- Find the bit rate of the source. Solution:

$$H(X) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{2}{16}\log 16 = \frac{15}{8} \text{ bit/sample}$$

$$R = H(X)\cdot f_s = H(X)\cdot 2F_{\max} = 15 \text{ kbit/s}$$

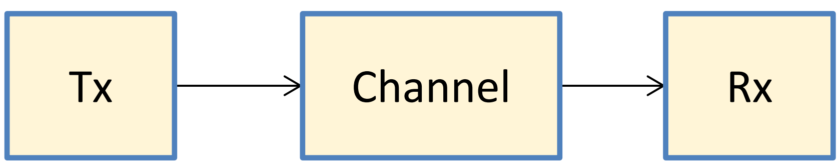
- What would be the answer for a uniformly-distributed source?

Channel Capacity

- This is the most fundamental notion in communication & information theory. It gives the fundamental limit on reliable communications over a given channel.
- Channel capacity C: maximum rate [bit/s] of reliable transmission of information over that channel.
- Reliable (error-free) transmission is possible only if

$$R_b < C$$

- The limit comes from the laws of Nature, not technological limitations.



Capacity of AWGN Channel

- Additive white Gaussian noise (AWGN) channel:

$$y = x + \xi$$

- where x – channel input, y – the output, ξ - AWG noise.
- Its capacity is (Shannon, 1948):

$$C = \Delta f \cdot \log(1 + SNR) \text{ [bit/s]}$$

- where SNR – signal-to-noise power ratio, Δf - channel bandwidth. Bandwidth can be traded for power and vise versa!
- Reliable (error-free) transmission is possible only if

$$R_b[bits/s] < C$$

- Modern systems can approach it closely.

Claude Shannon

The Bell System Technical Journal

Vol. XXVII *July, 1948* *No. 3*

A Mathematical Theory of Communication

By C. E. SHANNON

Introduction

THE recent development of various methods of modulation such as PCM and PPM which exchange bandwidth for signal-to-noise ratio has intensified the interest in a general theory of communication. A basis for such a theory is contained in the important papers of Nyquist[1] and Hartley[2] on this subject. In the present paper we will extend the theory to include a number of new factors, in particular the effect of noise in the channel, and the savings possible due to the statistical structure of the original message and due to the nature of the final destination of the information.

The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point. Frequently the messages have *meaning*; that is they refer to or are correlated according to some system with certain physical or conceptual entities. These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one *selected from a set* of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.

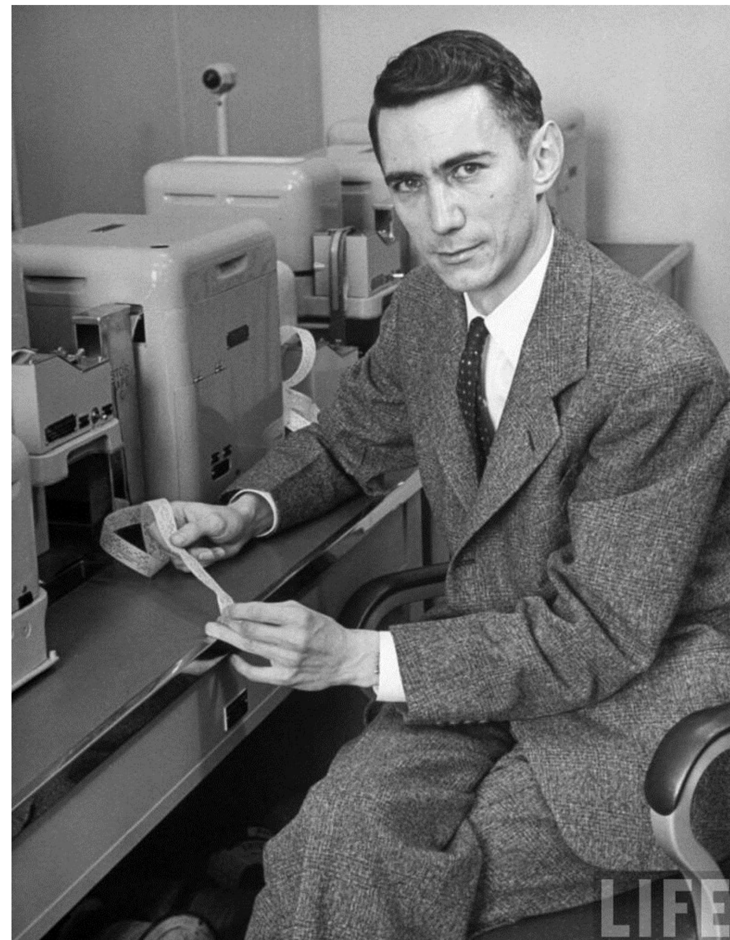
If the number of messages in the set is finite then this number or any monotonic function of this number can be regarded as a measure of the information produced when one message is chosen from the set, all choices being equally likely. As was pointed out by Hartley the most natural choice is the logarithmic function. Although this definition must be generalized considerably when we consider the influence of the statistics of the message and when we have a continuous range of messages, we will in all cases use an essentially logarithmic measure.

The logarithmic measure is more convenient for various reasons:

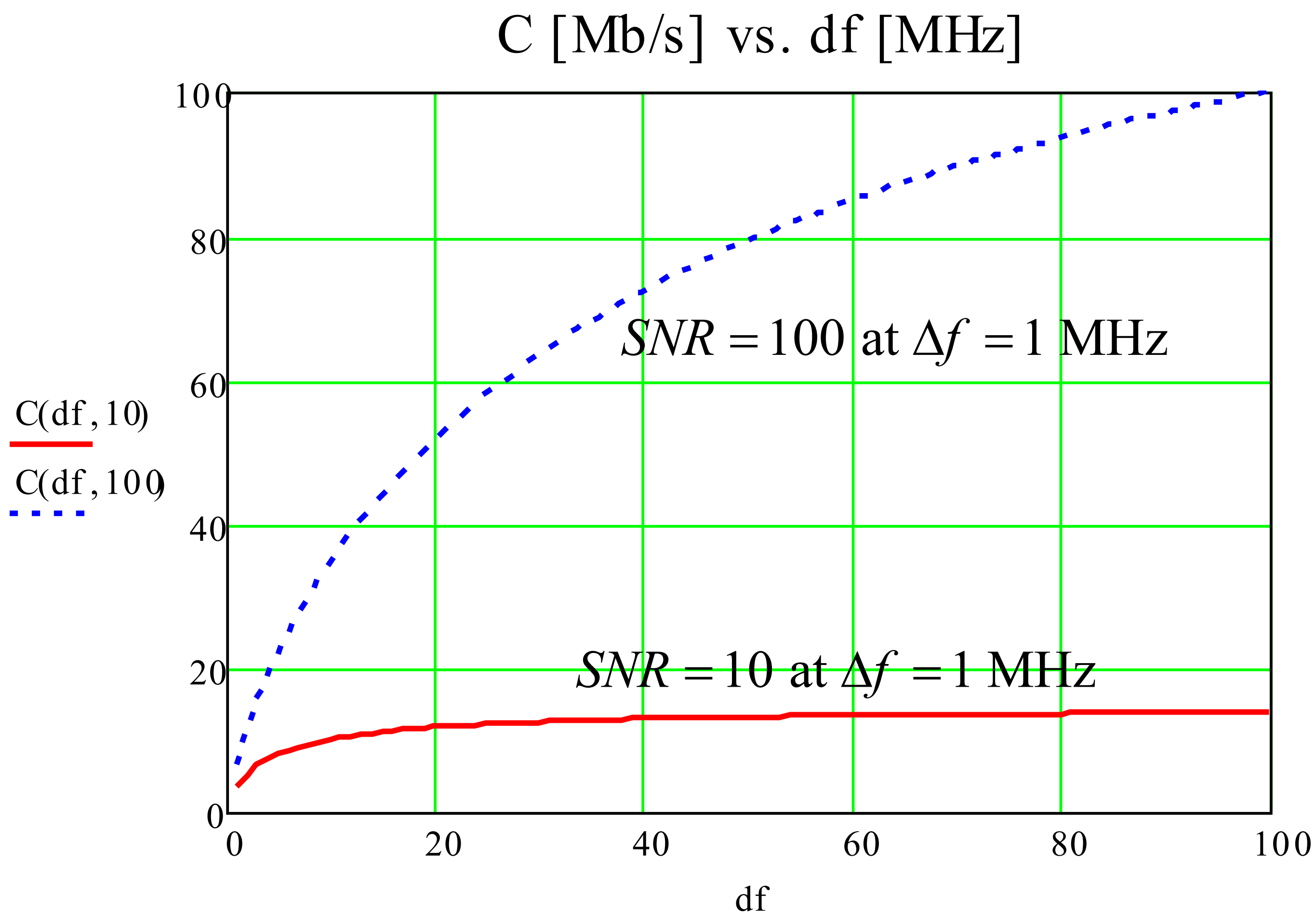
1. It is practically more useful. Parameters of engineering importance

[1] Nyquist, H., "Certain Factors Affecting Telegraph Speed," *Bell System Technical Journal*, April 1924, p. 324; "Certain Topics in Telegraph Transmission Theory," *A. I. E. E. Trans.*, v. 47, April 1928, p. 617.
[2] Hartley, R. V. L., "Transmission of Information," *Bell System Technical Journal*, July 1928, p. 535.

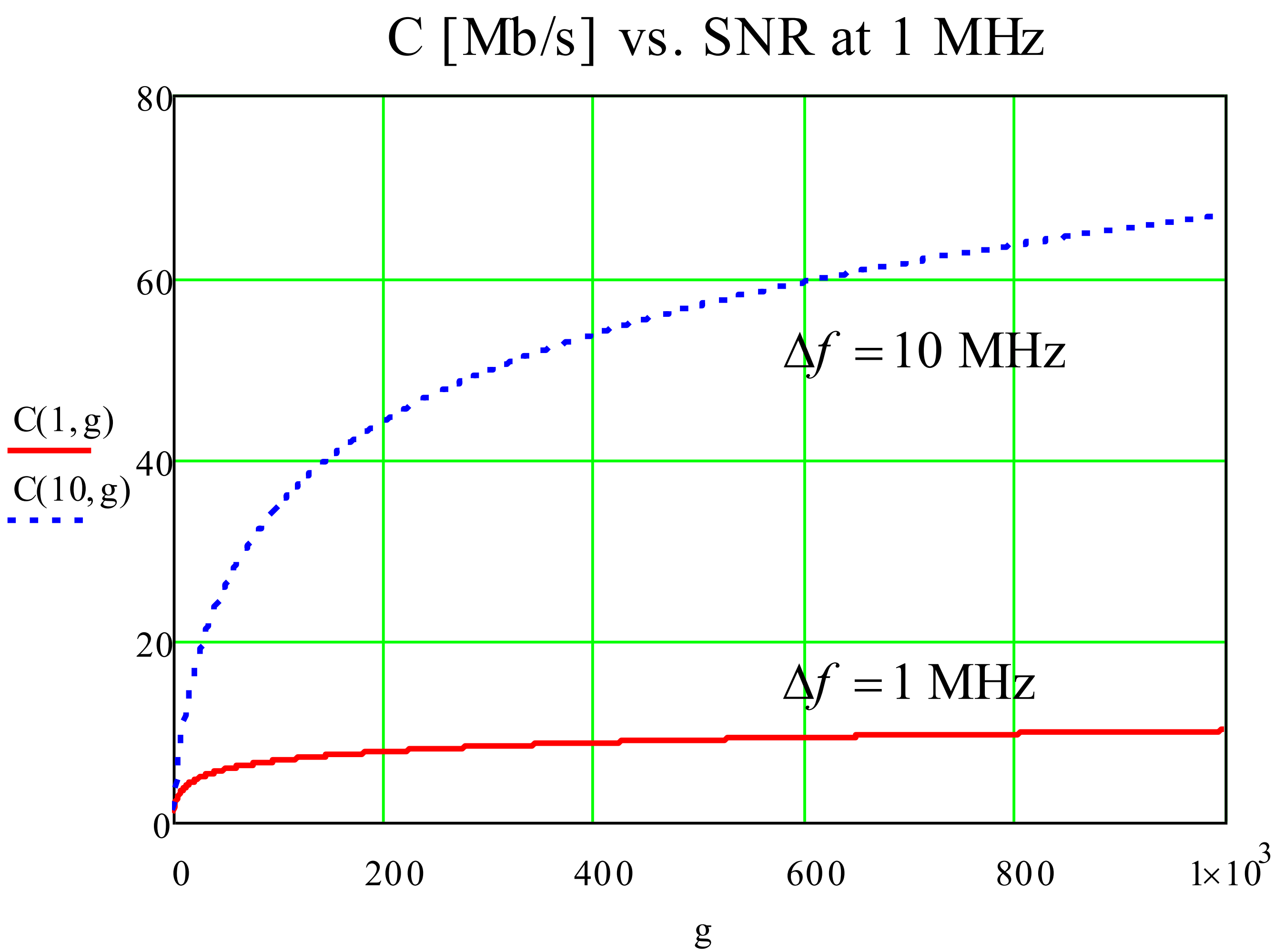
- Farther of Information Theory
- **Born**: 30. Apr. 1916, Michigan, USA
- **Died**: 24 Feb. 2001, Massachusetts, USA



Capacity of AWGN Channel



Capacity of AWGN Channel



Example: Telephone Channel

$$\boxed{\Delta f = 3.4 \text{ kHz}, \ SNR = 50 \text{ dB}}$$

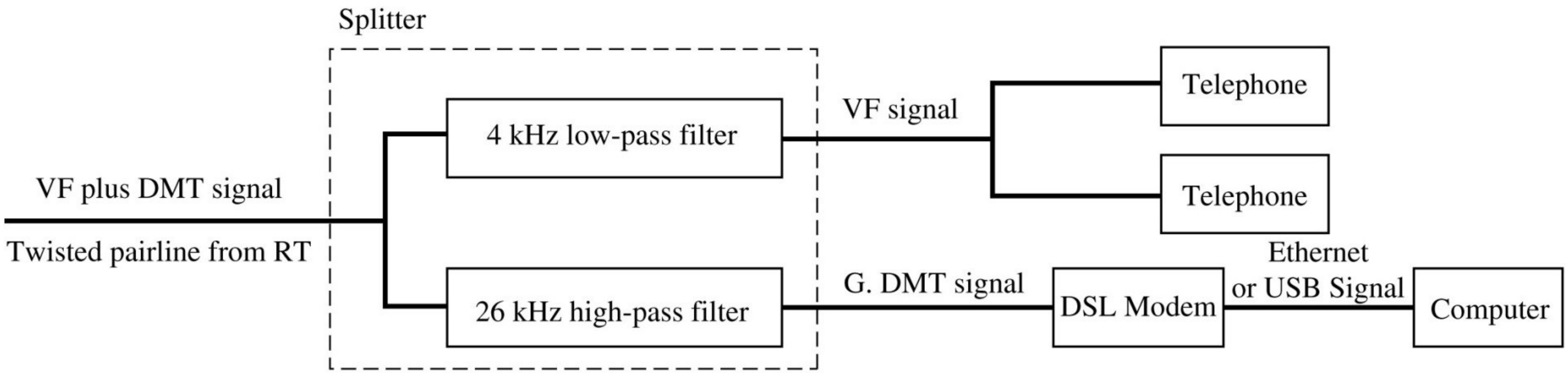
$$\Downarrow$$

$$\boxed{C = \Delta f \cdot \log(1 + SNR) \approx 56 \text{ kbit/s}}$$

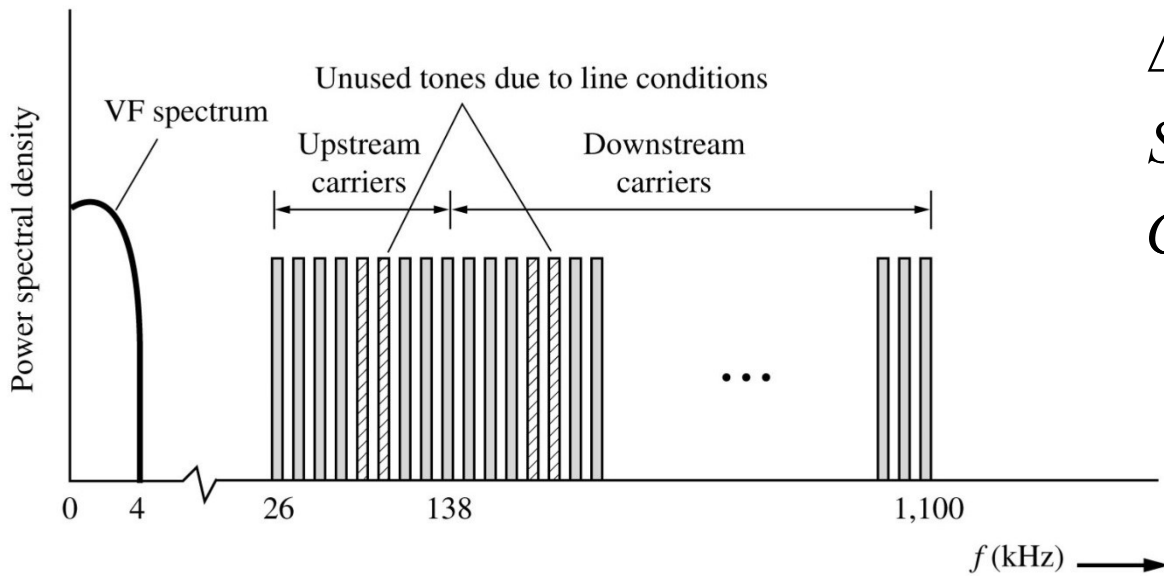
Linear SNR, not in dB!

How to increase C?

Example: Telephone Channel, DSL



(a) Customer Premises Equipment (CPE)



$$\Delta f \approx 1 \text{ MHz}$$

$$SNR = 50 \text{ dB}$$

$$C = ?$$

(b) VF plus DMT Spectrum

Couch, Digital and Analog Communication Systems, Seventh Edition ©2007 Pearson Education

Summary

- Information source. Simple model: discrete memoryless source.
- Quantitative measure of information. Entropy.
- Rate of a discrete memoryless source.
- Communication channel model.
- Channel capacity.
- Capacity of AWGN channel.

- **Homework**: Reading, Couch, 1.9, 1.10. J.G. Proakis, M. Salehi, Fundamentals of Communication Systems, 2014 (or 2005), Ch. 12.1, 12.2, 12.4 – 12.6.
- Attempt to solve some examples. Follow the topics we discussed in the class.
- Extra reference: T.M. Cover, J.A. Thomas, Elements of Information Theory, Wiley, 2006 (graduate-level textbook, good if you wish to know more about information theory and coding).