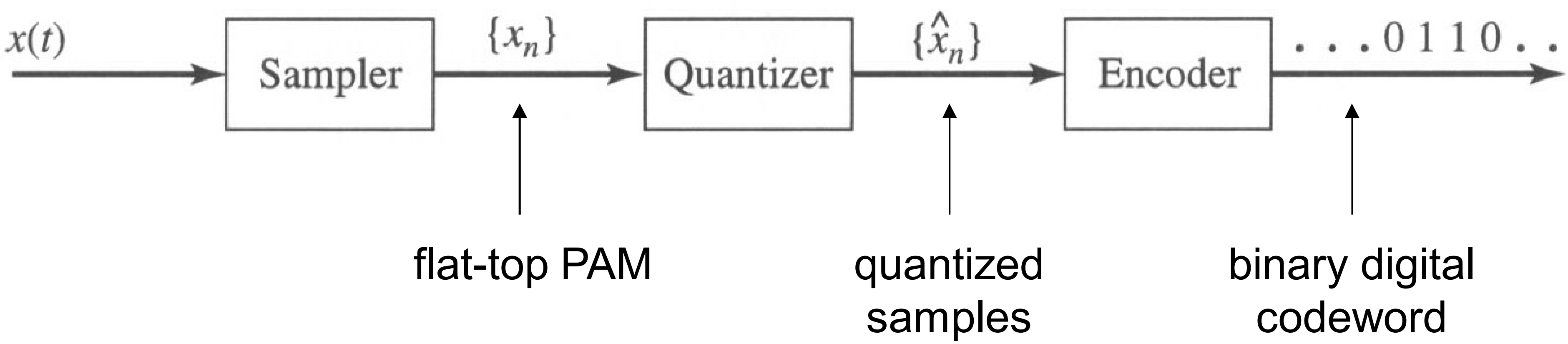
Pulse Code Modulation (PCM)

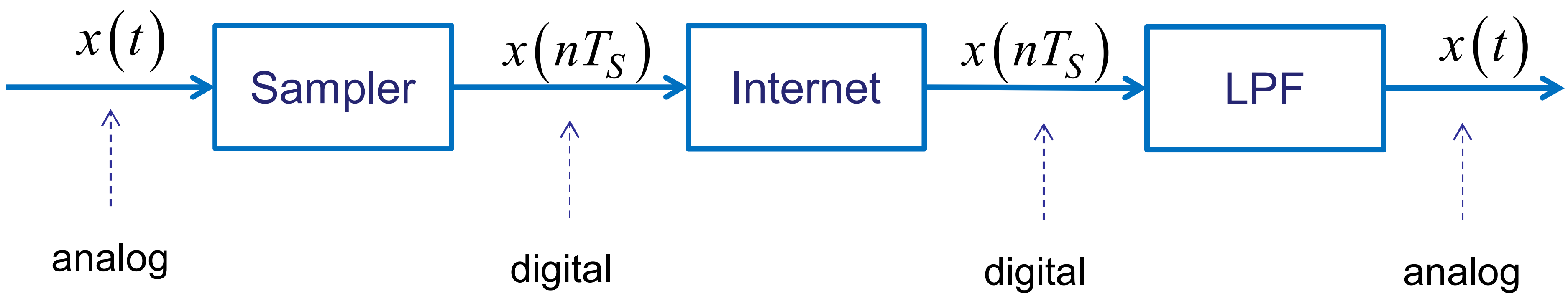
- PCM -> analog-to-digital (ADC) conversion. Instantaneous samples of an analog signal are represented by digital words in a serial bit stream.
- 3 main steps: *Sampling* (i.e., flat-top PAM), *Quantizing* (fixed number of levels is allowed), and *Encoding* (binary digital word).

Block Diagram of a PCM Modulator



Sampling Theorem and the Internet

(somewhat simplified: no quantizing yet)



$$\boxed{x(t)=\sum_{n=-\infty}^{\infty} x(nT_S)\,\mathrm{sinc}\left(\frac{t}{T_S}-n\right)}$$

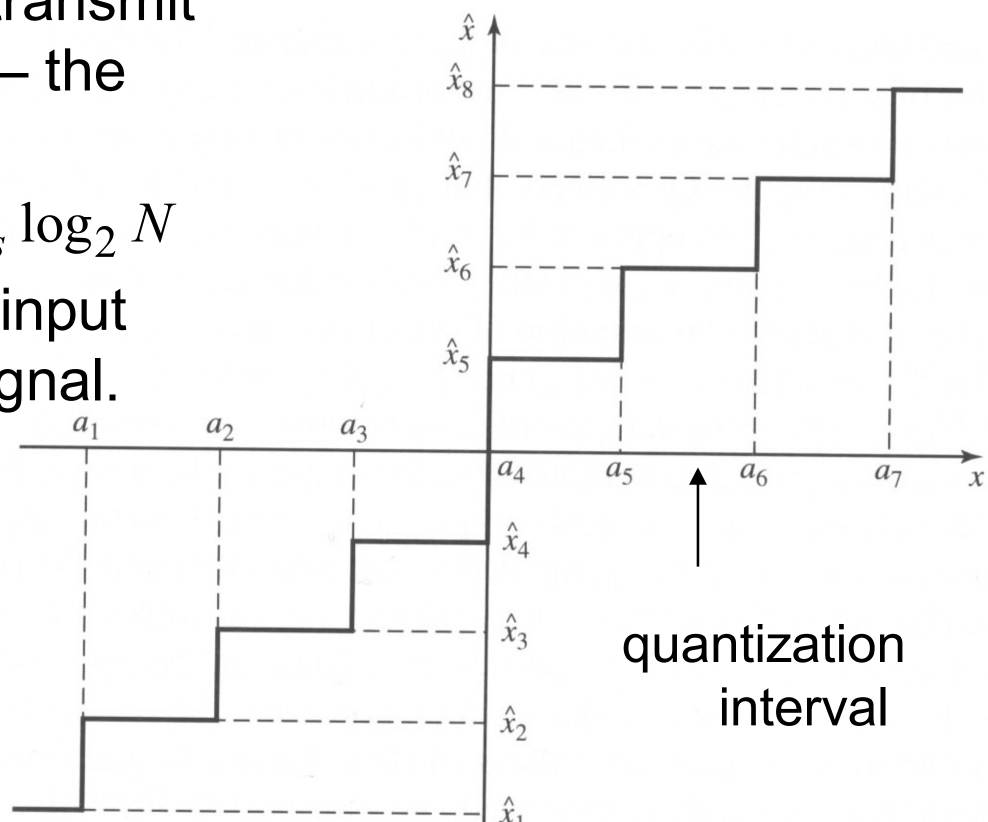
PCM Modulator

- Analog signal x(t) is bandlimited to F_{max} . If not, LPF is used (pre-sampling filter).
- The **sampling** is done at higher than Nyquist rate -> guard band, $f_s = 2F_{\max} + \Delta f$. Usually – flat-top PAM.
- **Quantizing**: the sample level is rounded off to the closest allowed level (only a fixed finite number of levels are allowed).
- **Encoding**: each allowed (quantized) level is represented by a (unique) binary code word.
- Serial transmission is used for binary digits. Thus, higher bandwidth is required.

Quantizing

- The exact sample value $x(nT_s)$ is replaced by the closest value allowed. The infinite number of levels is transformed into a finite number of levels.

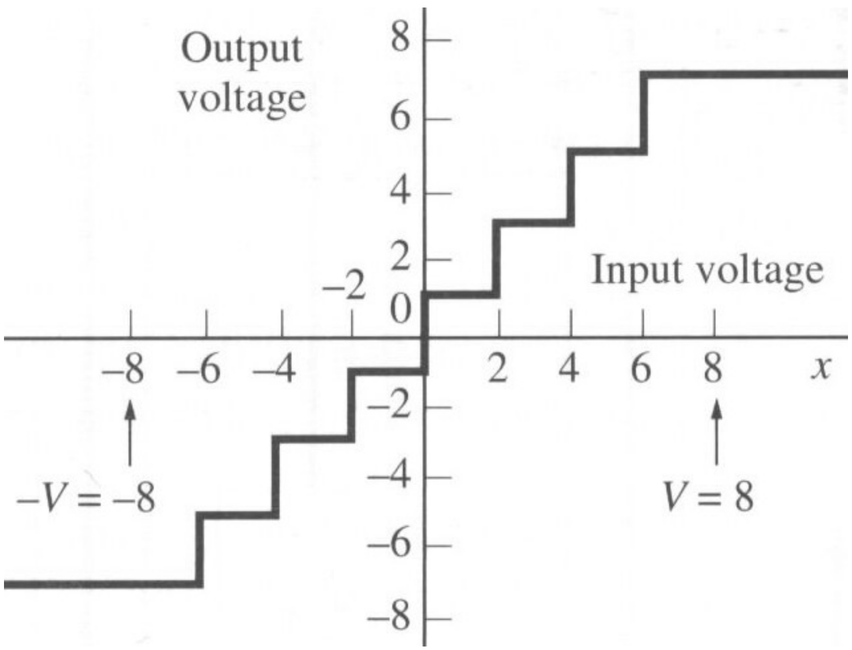
- Uniform quantizing: all steps are equal.
- The number of bits required to transmit each sample: $R_1 = \log_2 N$, N – the number of quantized levels.
- Transmission rate [bit/s]: $R = f_s \log_2 N$
- Example: 8-level quantizer. x – input analog signal, $\hat{x}_k$ - quantized signal. R_1=3 bits, $R = 3 f_s$



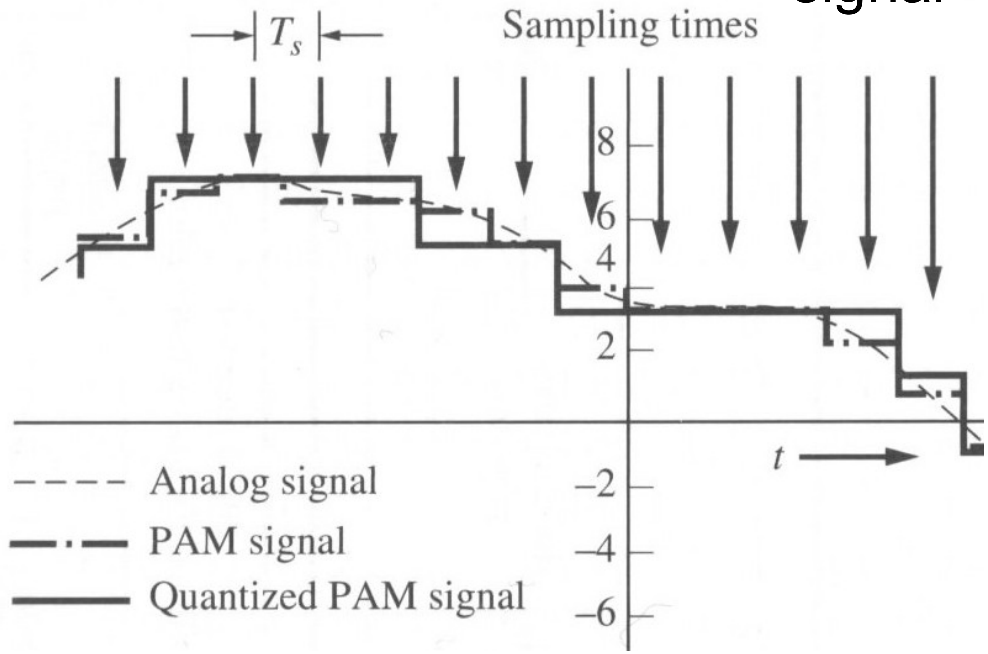
- Reversible ?

Quantizing: Example

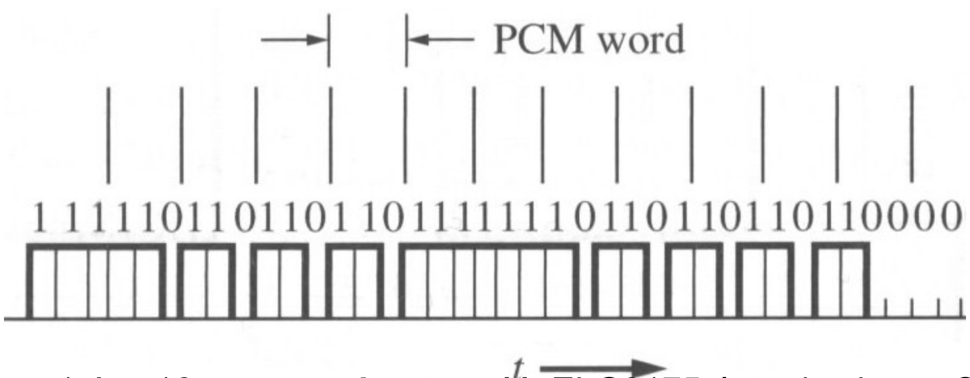
quantizer characteristic



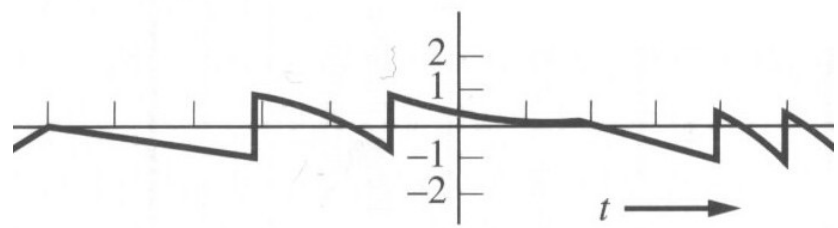
quantized signal



output PCM signal



quantization error (noise)



Example: sinusoid + binary quantizing.

Quantization Noise & SQNR

- The quantization function is noninvertible -> some information is lost. The effect is described using quantization noise.
- Mean square error (distance):

$$D = E\left[\left(x - Q(x)\right)^2\right] = \int_{-\infty}^{\infty} (\underbrace{x - \hat{x}}_{\varepsilon})^2 \rho_X(x)\,dx,\ \hat{x} = Q(x)$$

$$D = \frac{1}{N}\sum_i \int_{\Delta x_i} \left(x - \hat{x}\right)^2 \rho_i(x)\,dx,\ \rho_X(x)\text{ - pdf of } x$$

- Definition of quantization noise (error signal) power:

$$P_q = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} D(t)\,dt \xrightarrow{\text{const D}} D$$

- Signal to quantization noise ratio: $\boxed{SQNR = P_x / P_q}$
- Note: for random stationary x(t), the power is the variance:

$$P_x = E\left[x^2\right] = \int_{-\infty}^{\infty} x^2 \rho_X(x)\,dx$$

Uniform PCM

- The input signal range: $x \in [-x_{\max}, +x_{\max}]$
- All the quantization intervals are equal: $\Delta x_i = \Delta = 2x_{\max} / N$
- When N is large, Δ is small and the error $\varepsilon = x - Q(x)$ is uniformly distributed within $[-\Delta/2, +\Delta/2]$ for each quantization interval: $\rho_\varepsilon(\varepsilon) = 1/\Delta$
- Quantization noise power is

$$P_q = D = \frac{1}{N} \sum_i \int_{\Delta x_i} \varepsilon^2 \rho_\varepsilon(\varepsilon) d\varepsilon = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \varepsilon^2 d\varepsilon = \frac{\Delta^2}{12} = \frac{x_{\max}^2}{3N^2}$$

- SQNR is $SQNR = \dfrac{P_x}{P_q} = \dfrac{3N^2 P_x}{x_{\max}^2} \le 3N^2$

peak SQNR

- Peak factor $\beta = \dfrac{x_{\max}^2}{P_x} \ge 1 \Rightarrow \boxed{SQNR = \dfrac{3N^2}{\beta} \le 3N^2}$

Uniform PCM & SQNR

- Log form of the uniform SQNR law (6 dB law):

$$\boxed{SQNR\big|_{dB} \approx -\beta\big|_{dB} + 6\nu + 4.8}$$

- where $N = 2^{\nu},\ \nu$ - the number of bits.
- Each extra bit adds 6 dB to SQNR.
- **Example**: $x \in [-1, +1]$, uniform PCM with 256 levels. Find SQNR.
- $\nu = \log N = 8,\ P_x = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{3},\ \beta = \frac{x_{\max}^2}{P_x} = 3$

$$SQNR = 3N^2/\beta \approx 6.6 \cdot 10^4 \approx 48 \text{ dB}$$

- Homework: do the same for x_{max}=2 and $x \in [-1, +1]$. Compare with the result above and make conclusions.

Bandwidth of PCM

- If using rectangular pulses, absolute bandwidth is infinite. Power bandwidth is finite.
- Non-rectangular ("rounded") pulses may be used to transmit digital codewords (110100..), which are bandlimited.
- Fundamental limit is obtained using the sampling theorem. The minimum number of samples for a perfect reconstruction of a bandlimited signal is f_s /second. If N quantization levels are used, then

$$\boxed{R = f_s \log_2 N} \xrightarrow{\text{Nyquist}} \boxed{R = 2F_{\max} \log_2 N \ \text{[bit/s]}}$$

- The minimum bandwidth to transmit R bits/s using binary mod. is R/2 (sampling theorem again! See Lec. 12 for more details). Hence,

$$\boxed{\Delta f_{\min} = \frac{1}{2} f_s \log_2 N} \xrightarrow{\text{Nyquist}} \boxed{\Delta f_{\min} = F_{\max} \log_2 N}$$

Example: PCM for Telephone System

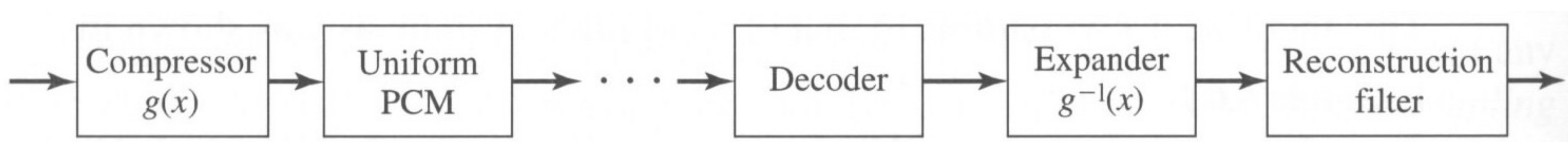
- Telephone spectrum: [300 Hz, 3400 Hz]
- Min. sampling frequency: $f_{s,\min} = 2F_{\max} = 6.8 \text{ kHz}$ (or [sam./s])
- Some guard band is required: $f_s = 2F_{\max} + \Delta f_g = 8 \text{ kHz}$
- 8-bit codewords are used -> N=256.
- The transmission rate: $R = f_s \nu = 64 \text{ kbit/s}$
- Minimum absolute bandwidth: $\Delta f_{\min} = R/2 = 32 \text{ kHz}$
- Peak SQNR:

$$SQNR = 3N^2 \approx 2 \cdot 10^5 \approx 53 \text{ dB}$$

- Another example: CD player (see the text by Proakis and Salehi (2nd ed.), section 6.8).

Nonuniform PCM

- Uniform PCM is good for uniform signal distributions, but not efficient for nonuniform ones.
- Example: speech signal has large probability of small values and small prob. of large ones.
- Solution: allocate more levels for small amplitudes and less for large. Total quantizing noise is greatly reduced (see equations above).
- Typical solution for nonuniform PCM modulator: compress signal first, then apply uniform PCM. Rx end: demodulate uniform PCM and expand it. The technique is called companding.



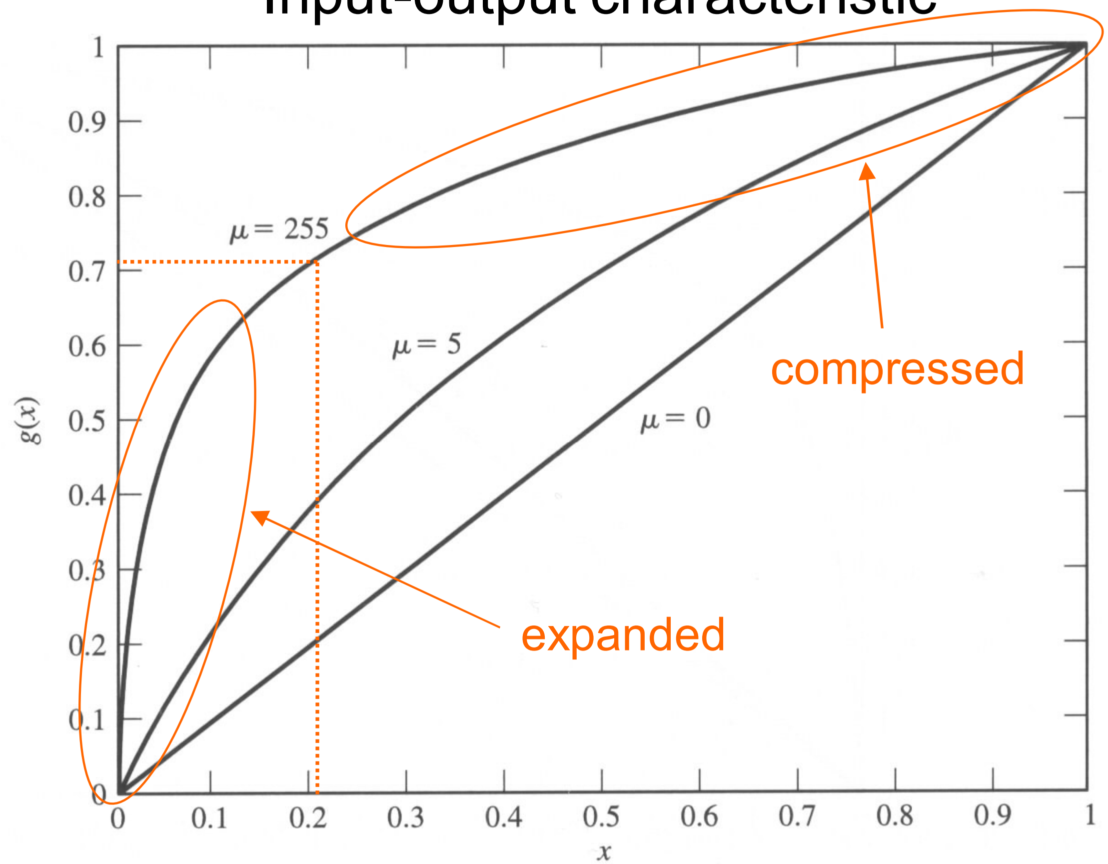
μ-Law Companding (Speech)

- Logarithmic function is used, $|x| \le 1$, where μ controls the amount of compression.

$$g(x) = \frac{\log(1+\mu|x|)}{\log(1+\mu)} \operatorname{sgn}(x)$$

- Used in US & Canada ($\mu = 255$), + a uniform 128 levels (7 bits) quantizer.
- The compander improves SQNR by approx. 24 dB.

Input-output characteristic

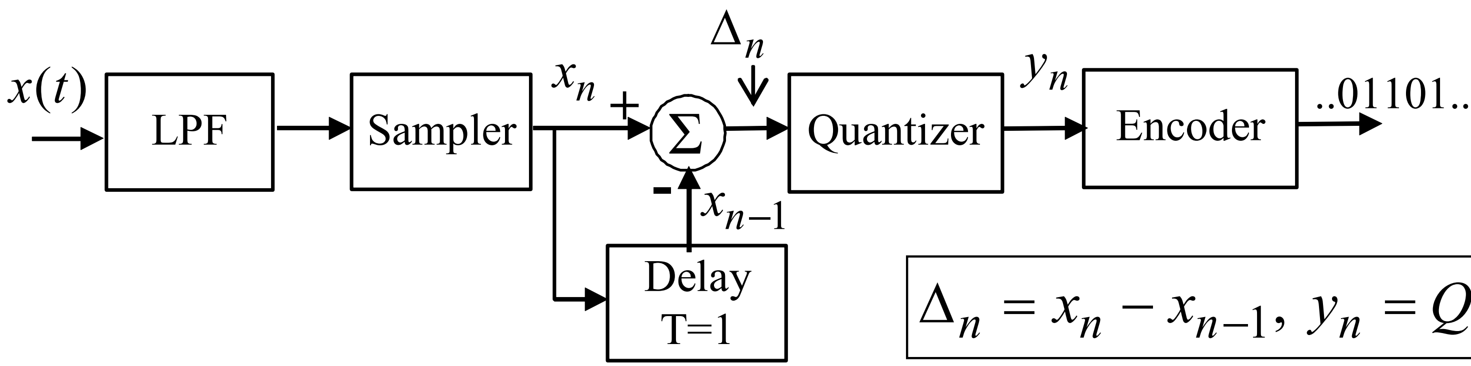


Differential PCM

- Samples of a bandlimited signal are correlated -> previous sample gives information about the next one. Example: if previous samples are small, the next one will be small with high probability.
- This can be used to improve PCM performance: to decrease the number of bits used (and, hence, the bandwidth) or to increase SQNR for a given bandwidth.
- Main idea: quantize and transmit the difference between two adjacent samples rather than sample values.
- Since two adjacent samples are correlated (bandlimited signal!), their difference is small and requires less bits to transmit.

Simple DPCM System

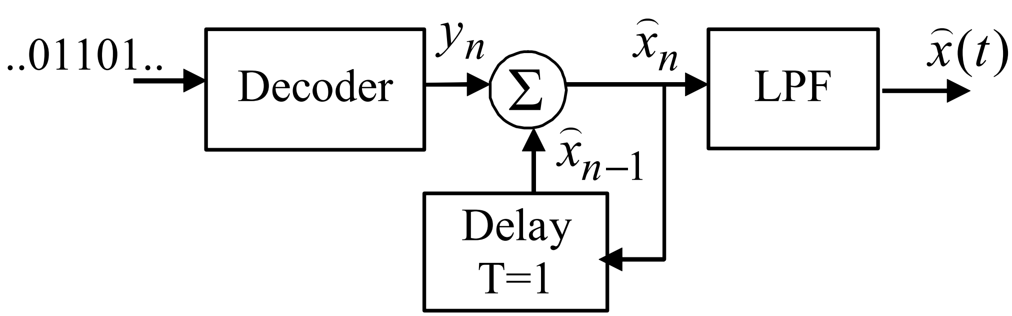
Modulator



$$\Delta_n = x_n - x_{n-1},\ y_n = Q[\Delta_n] = \Delta_n + \varepsilon_n$$

$$\hat{x}_n = \sum_{k=0}^{n} y_k = \sum_{k=0}^{n} \Delta_k + \sum_{k=0}^{n} \varepsilon_k$$
$$= y_n + \hat{x}_{n-1} \rightarrow y_n = \hat{x}_n - \hat{x}_{n-1}$$

Demodulator



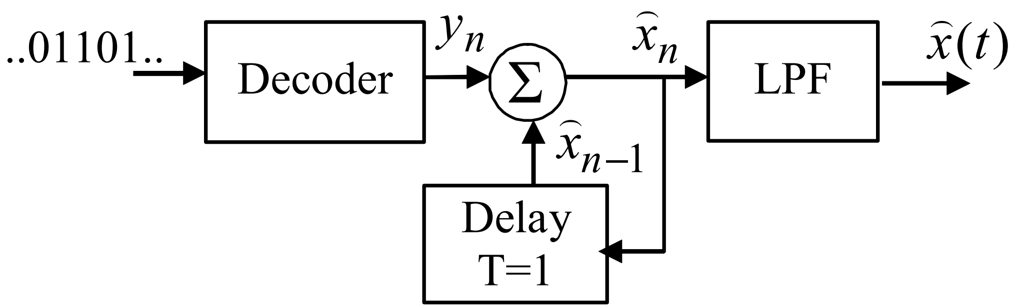
Very good quantization: $y_n \approx \Delta_n$

Hence, $\hat{x}_n$ and x_n satisfy the same difference equation -> must be the same!

Problem: quantization noise accumulation.

Quantization Noise Accumulation

Demodulator



$$\hat{x}_n = \sum_{k=0}^{n} y_k = \overbrace{\sum_{k=0}^{n} \Delta_k}^{x_n} + \sum_{k=0}^{n} \varepsilon_k$$

$$\Delta x_n = \hat{x}_n - x_n = \sum_{k=0}^{n} \varepsilon_k$$

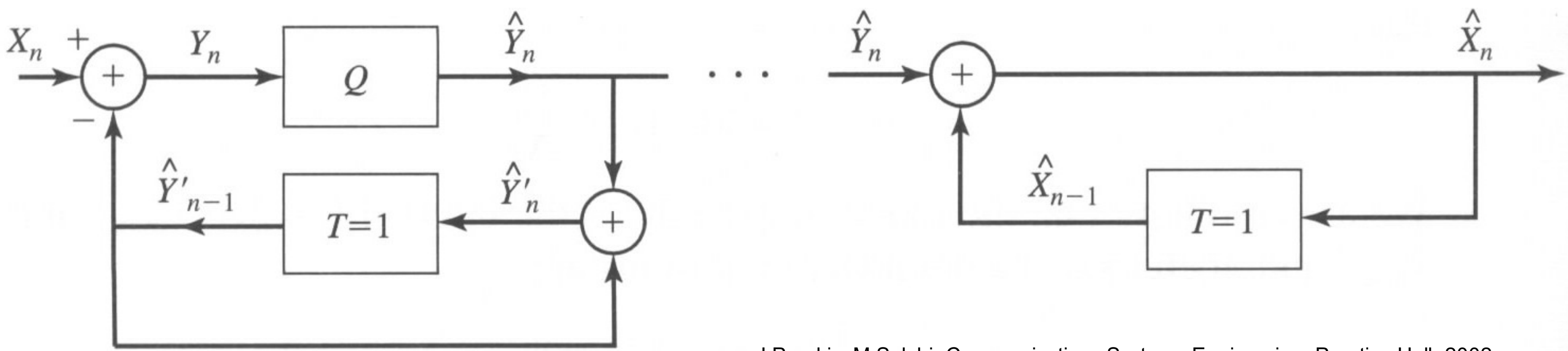
← Includes past Q. noise contributions!

$$P\{\Delta x_n\} = \overline{|\Delta x_n|^2} = \sum_{k=0}^{n} \overline{|\varepsilon_k|^2} = \sum_{k=0}^{n} P\{\varepsilon_k\}$$

← Noise power is always added, never subtracted! (assuming independence)

Improved DPCM System

No quantization noise accumulation.



J.Proakis, M.Salehi, Communications Systems Engineering, Prentice Hall, 2002

- Analysis:

$$\begin{cases} Y_n = X_n - \hat{Y}'_{n-1} \\ \hat{Y}'_n = \hat{Y}_n + \hat{Y}'_{n-1} \end{cases}$$

very good quantizer

$$Y_n = X_n - \hat{Y}'_{n-1} \approx \hat{Y}_n = \hat{Y}'_n - \hat{Y}'_{n-1} \rightarrow X_n \approx \hat{Y}_n'$$

$$\hat{X}_n = \hat{Y}_n + \hat{X}_{n-1}$$

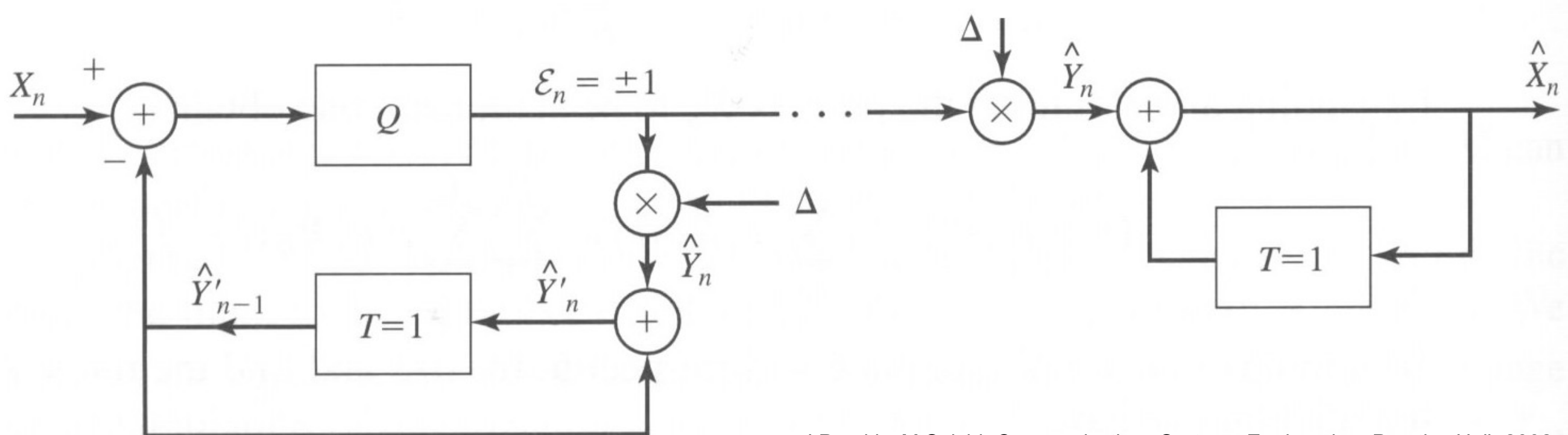
$$\hat{X}_n = \sum_{i=0}^{n} \hat{Y}_i$$

$\hat{X}_n$ and $\hat{Y}'_n$ satisfy the same difference equation -> must be the same -> $\hat{X}_n = \hat{Y}'_n \approx X_n$

In general, $\boxed{\varepsilon_n = \hat{Y}_n - Y_n = \hat{X}_n - X_n}$

Delta Modulation

- This is a simplified version of DPCM. A 1-bit, 2 level -> $\pm\Delta$ quantizer is used.

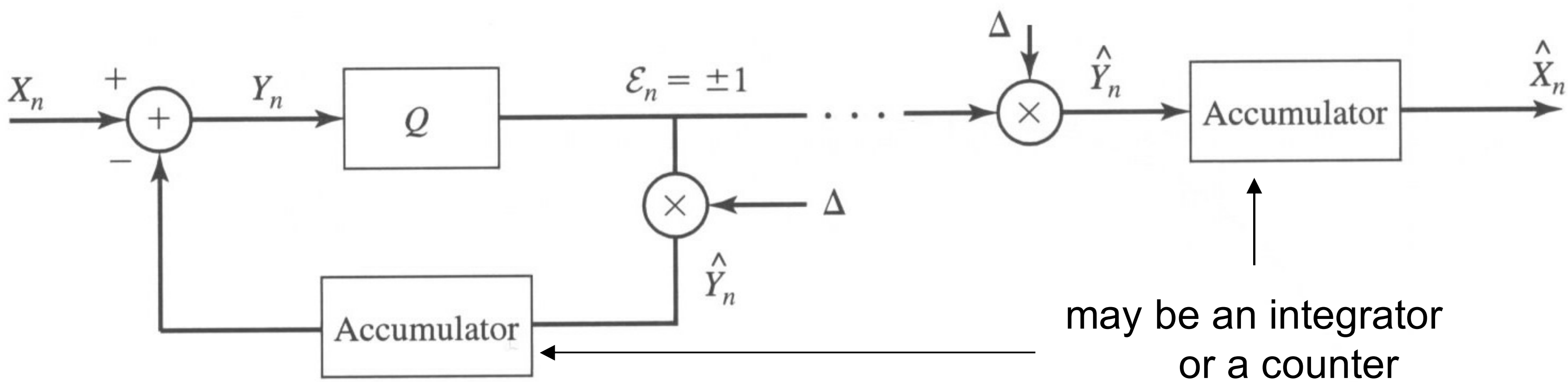


J.Proakis, M.Salehi, Communications Systems Engineering, Prentice Hall, 2002

- Since there are only 2 levels, the Y_n dynamic range must be low to keep quantization noise low.
- This, in turn, means that X_n and X_{n-1} must be highly correlated -> sampling frequency must be much higher than the Nyquist rate.

Delta Modulation

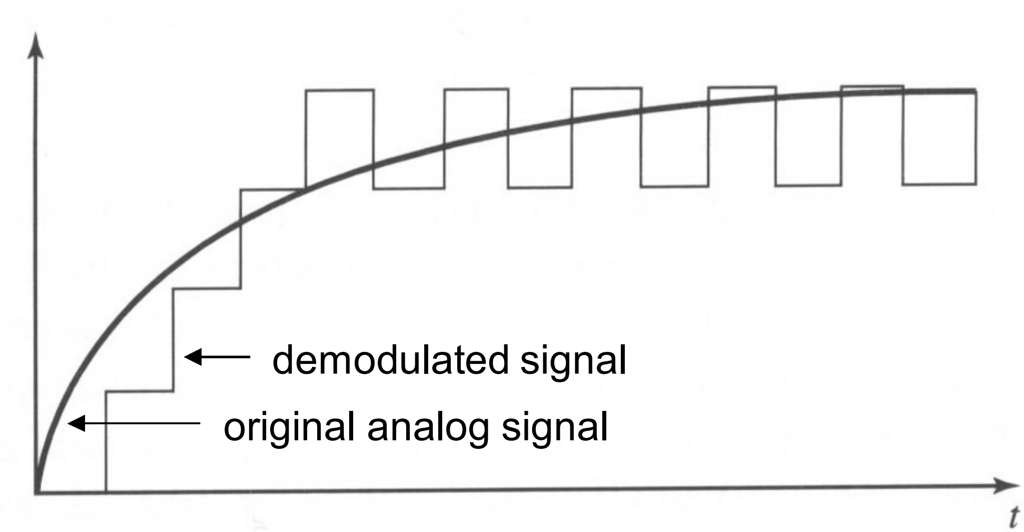
- Despite of the high sampling frequency, transmission rate is low (less than for PCM) because there is only 1 bit/sample to transmit.
- Major advantage -> simple structure.



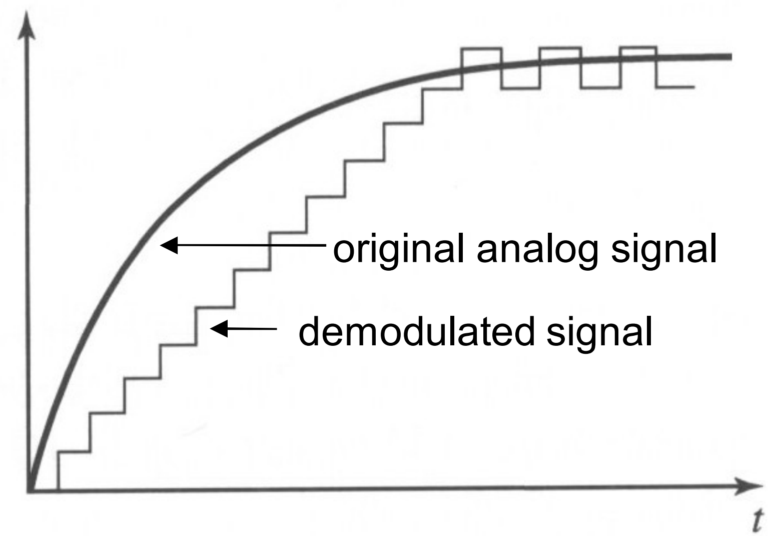
- Major disadvantage: granular noise and slope-overload distortion

Granular Noise and Slope-Overload Distortion

large Δ -> granular noise



small Δ -> slope-overload distortion



- Step size is very important.
- Small step size results in slope-overload distortion.
- Large step size results in granular noise.
- Solution: adaptive delta-modulation.

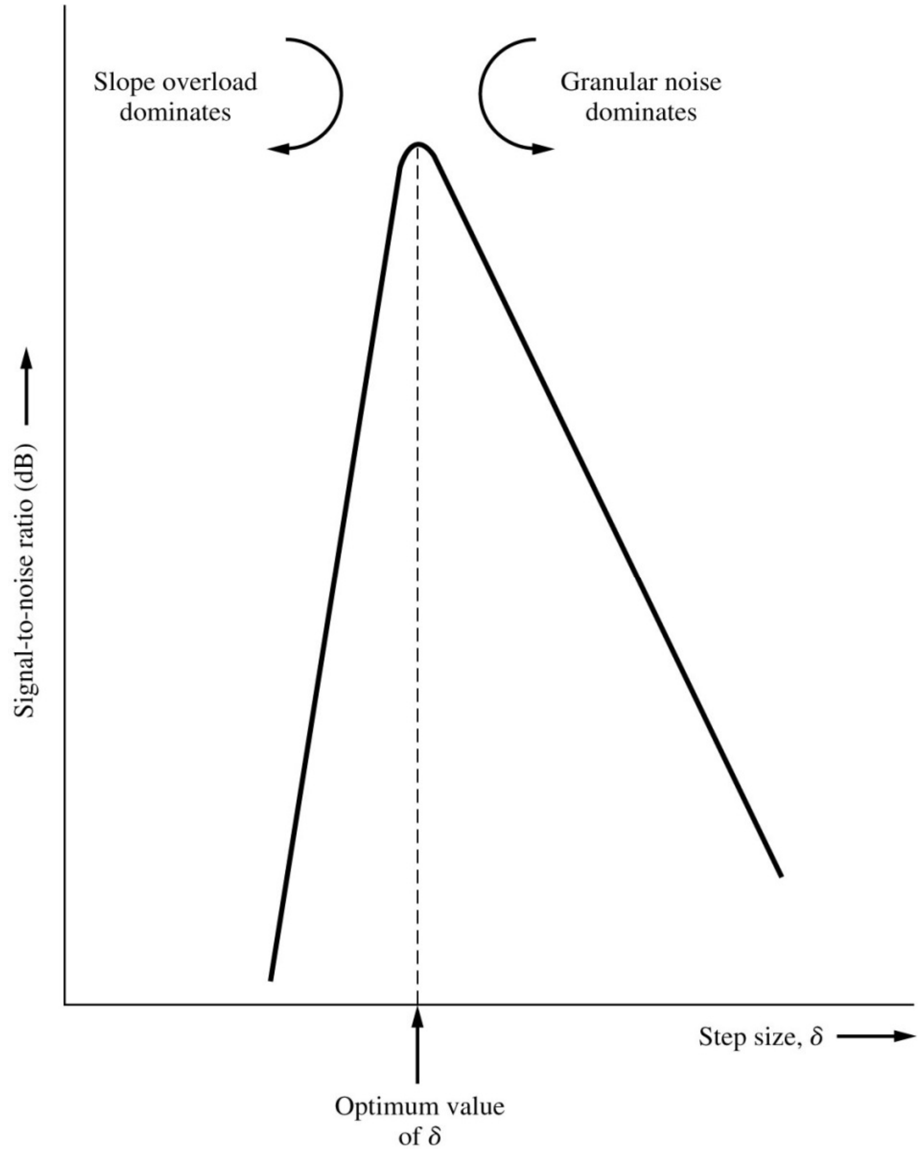
Example: DM for Sinusoidal Signal

- Maximum slope generated by DM demodulator is $$s_m = \Delta / T_s = \Delta \cdot f_s$$
- For a sinusoidal input, the slope is $$s_{in} = \frac{d}{dt} x(t) = A\omega_{in} \cos \omega_{in} t$$
- The maximum input slope is $$s_{in,\max} = A\omega_{in}$$
- No slope overload distortion if $$s_m \geq s_{in,\max} \rightarrow \Delta \geq 2\pi A f_{in} / f_s$$
- SQNR if no overload distortion (see the text): $$SQNR = \frac{3}{8\pi^2} \frac{f_s^3}{f_{in}^2 f_{LPF}}$$

- Example: $x(t) = \sin 2\pi 10^3 t$, $f_s = 10$ kHz, $f_{LPF} = 2$ kHz

$$\Delta \geq 2\pi 10^3 / 10^4 \approx 0.6, \; SQNR \approx 13 \text{ dB}$$

Granular Noise and Slope-Overload Distortion

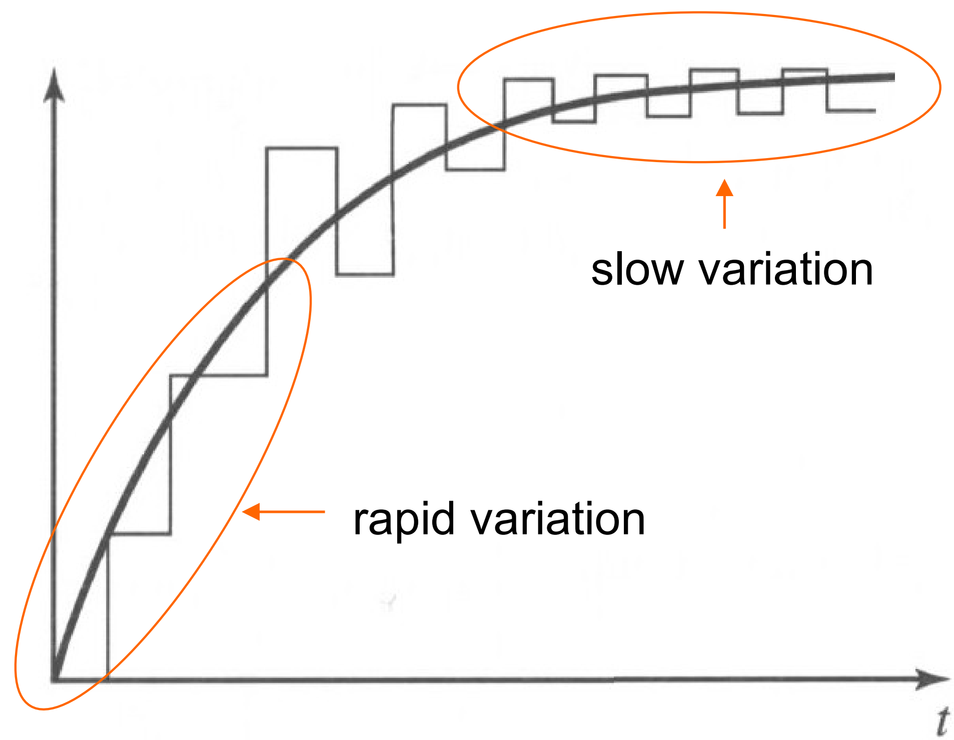


Couch, Digital and Analog Communication Systems, Pearson Education.

Adaptive Delta-Modulation

- Main idea: change step size according to changes in the input signal.
- If the input changes rapidly -> large step size. If the input changes slowly -> small step size.

- How to implement step size change?
- Simple solution: if two successive outputs have the same sign -> increase step size; if they are of opposite sign -> decrease step size.



Summary

- Pulse code modulation (PCM): Sampling, quantizing and encoding.
- Uniform quantizing. SQNR. Nonuniform quantizing.
- Differential PCM. Block diagrams (simple and improved). Quantization noise accumulation.
- Delta modulation. Block diagrams.
- Granular noise and slope-overload distortion. Limitation on the step size.
- Comparison of PCM and delta modulation.
- Adaptive delta-modulation.

- **Homework**: Reading: Couch, 3.1-3.3, 3.7, 3.8. Study carefully all the examples, make sure you understand them and can solve with the book closed.