ELG7177: MIMO Comunications

Lecture 8

Dr. Sergey Loyka
EECS, University of Ottawa

February 13, 2019

Multi-User Systems

- Can multiple antennas offer advantages for multi-user systems?
- How to realize those advantages?
- Channel/system models
- Optimal Tx/Rx strategies

Multi-User System: FDMA

- multiple (N) users communicate to a single base station (BS)
- consider 1st FDMA
- Δf = per-user bandwidth
- $\Delta F = N\Delta f$ = total system bandwidth
- $C_1 = \Delta f \log(1+\gamma)$ = per-user capacity
- $C_s = NC_1 = \Delta F \log(1+\gamma)$ = total system capacity

- **Can we do better ?**

Multi-User System: FDMA

- try antenna array at the BS: $\gamma \rightarrow m\gamma$,

$$C_{1a} = \Delta f \log(1 + m\gamma), \; C_{sa} = NC_{1a} = \Delta F \log(1 + m\gamma) \qquad (1)$$

- improvement via the SNR gain m, but only logarithmic in m -> not much
- **Can we do better ???**

SDMA via Null Forming

With $m \geq N$, the BS can receive(transmit) user 1 signal while nulling out all other $(N-1)$ users.

Hence, each user can use the total (aggregate) system bandwidth $\Delta F = N\Delta f$ instead of Δf:

$$C_1' = \Delta F \log(1+\gamma) = NC_1 = C_s \gg C_1, \quad (2)$$

$$C_s' = NC_1' = N\Delta F \log(1+\gamma) = NC_s \gg C_s \quad (3)$$

Much better for $N \gg 1$!

This is known as SDMA.

SDMA vs. FDMA: An Example

Example: $m = N = 10,\ \Delta f = 1$ MHz, $\gamma = 10$:

$$\begin{aligned} \text{FDMA}:\ & C_1 \approx 3\text{Mb/s},\ C_s \approx 30\text{Mb/s}, \\ \text{SDMA}:\ & C_1' \approx 30\text{Mb/s},\ C_s' \approx 300\text{Mb/s} \end{aligned}$$

Q1: How much more SNR do you need to go from C_1 to C_1' using just FDMA (with same bandwidth Δf)?

This is the antenna gain in a multi-user system.

Q2: evaluate $C_{1a},\ C_{sa}$ for this example and compare it to $C_1',\ C_s'$. Comment on the difference.

SDMA via Null Forming

The Rx signal at the BS is

$$\mathbf{y} = \sum_{k=1}^{N} \mathbf{h}_k x_k + \boldsymbol{\xi} \qquad (4)$$

To detect user 1, the BS nulls out (known as ZF) all other users:

$$y_{r1} = \mathbf{w}_1^{+} \mathbf{y} = \mathbf{w}_1^{+} \mathbf{h}_1 x_1 + \mathbf{w}_1^{+} \boldsymbol{\xi} \qquad (5)$$

$\mathbf{w}_1$ is the beamforming vector such that

$$\mathbf{w}_1 \perp \mathbf{h}_2, .., \mathbf{h}_N, \text{ i.e. } \mathbf{w}_1^{+} \mathbf{h}_k = 0, \ k = 2, .., N \qquad (6)$$

which is possible if $\mathbf{h}_1 \notin span\{\mathbf{h}_2, .., \mathbf{h}_N\}$.

SDMA via Null Forming

The process is repeated for all users.

It works if $\mathbf{H} = [\mathbf{h}_1, .., \mathbf{h}_N]$ is full-rank, i.e. all columns of $\mathbf{H}$ are linearly independent.

Q1: what happens if they are not? Why?

Q2: find the system capacity of SDMA via null forming.

Q3: **can we do better???**

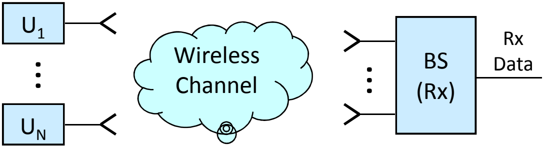
Multiple Access Channel (MAC)

Channel of the form

$$\mathbf{y} = \sum_{k=1}^{N} \mathbf{h}_k x_k + \boldsymbol{\xi} \qquad (7)$$

is known as multiple-access channel (MAC), or "uplink" (users-to-BS).

K single-antenna users communicate to a single multi-antenna BS:



Multiple Access Channel (MAC)

The capacity of MAC:

$$C = \max_{\mathbf{R}_x} \log \left| \mathbf{I} + \mathbf{H}\mathbf{R}_x\mathbf{H}^+ \right| \text{ s.t. } \mathbf{R}_x \geq 0,\ \mathbf{R}_x = \text{diag},\ r_{ii} \leq P_i \qquad (8)$$

and an optimal input is $X \sim CN(0, \mathbf{R}_x)$.

Multiple Access Channel (MAC)

The capacity of MAC:

$$C = \max_{\mathbf{R}_x} \log |\mathbf{I} + \mathbf{H}\mathbf{R}_x\mathbf{H}^+| \text{ s.t. } \mathbf{R}_x \geq 0,\ \mathbf{R}_x = \text{diag},\ r_{ii} \leq P_i \quad (8)$$

and an optimal input is $X \sim CN(0, \mathbf{R}_x)$.

But: **How to find the max???**

Multiple Access Channel (MAC)

The capacity of MAC:

$$C = \max_{\mathbf{R}_x} \log \left|\mathbf{I} + \mathbf{H}\mathbf{R}_x\mathbf{H}^+\right| \text{ s.t. } \mathbf{R}_x \geq 0,\ \mathbf{R}_x = \text{diag},\ r_{ii} \leq P_i \qquad (8)$$

and an optimal input is $X \sim CN(0, \mathbf{R}_x)$.

But: **How to find the max???**

Since $\mathbf{R}_x \leq \mathbf{P} = \text{diag}\{P_1, .., P_2\}$, it follows that

$$\mathbf{R}_x^* = \mathbf{P} \qquad (9)$$

i.e. transmission with full per-user power is optimal.

Multiple Access Channel (MAC)

Thus, the capacity of this MAC is

$$C = \log|\mathbf{I} + \mathbf{WP}| = \log|\mathbf{I} + \sum_k P_k \mathbf{h}_k \mathbf{h}_k^+| \tag{10}$$

and an optimal input is $X \sim CN(0, \mathbf{P})$.

Q1: compare this to the MIMO channel capacity (with the same $\mathbf{W} = \mathbf{H}^+\mathbf{H}$), which is better? Why?

Q2: compare this to the capacity of ZF SDMA, which is better?

Q3: is an optimal input unique? Explain.

An Example: Free-Space MAC

1. Consider a free-space MAC, $h_{ij} = 1$ for all i, j, with the same per-user powers, $P_i = P$. Find its capacity. Compare it with the ZF SDMA capacity.
2. Do the same for an orthogonal MAC, $\mathbf{H} = \mathbf{I}$, compare to $\#1$, make conclusions.

Capacity Region of MAC

What rates are achievable for each user?

Capacity region $\mathcal{C}$: a set of all simultaneously-achievable rates $(R_1, .., R_N)$.

How to characterize?

2-User MAC

Channel model: SISO channel for each user,

$$y(t) = x_1(t) + x_2(t) + \xi(t) \tag{11}$$

Capacity region $\mathcal{C}$: a set of all (R_1, R_2) satisfying

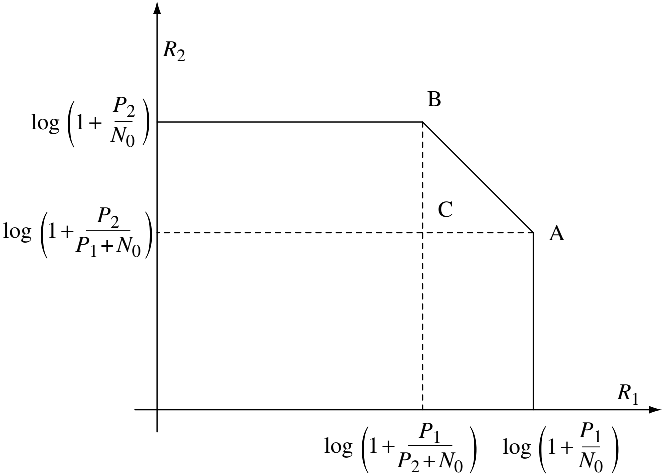
$$R_1 < \log\left(1 + \frac{P_1}{\sigma_0^2}\right) = C_1 \tag{12}$$

$$R_2 < \log\left(1 + \frac{P_2}{\sigma_0^2}\right) = C_2 \tag{13}$$

$$R_1 + R_2 < \log\left(1 + \frac{P_1 + P_2}{\sigma_0^2}\right) \tag{14}$$

P_k = user k power constraint.

2-User MAC: Capacity Region[1]



[1] D. Tse, P. Viswanath, Fundamentals of Wireless Communications, Cambridge University Press, 2005.

2-User MAC

Symmetric capacity: the largest common rate,

$$C_{sym} = \max_{(R,R)\in\mathcal{C}} R = ? \tag{15}$$

Sum capacity: the largest total rate

$$C_{sum} = \max_{(R_1,R_2)\in\mathcal{C}} R_1 + R_2 = ? \tag{16}$$

Q: evaluate these capacities for the 2-user MAC above.

Successive Interference Cancellation (SIC) Rx

MAC Rx: How to recover individual (per-user) data?

Successive Interference Cancellation (SIC) Rx

MAC Rx: How to recover individual (per-user) data?

1. detect user 1 treating user 2 as interference:

$$y = x_1 + x_2 + \xi \to R_1 < \log\left(1 + \frac{P_1}{P_2 + \sigma_0^2}\right) \quad (17)$$

2. subtract detected signal:

$$y' = y - x_1 = x_2 + \xi \quad (18)$$

3. detect user 2:

$$R_2 < \log\left(1 + \frac{P_2}{\sigma_0^2}\right) \quad (19)$$

so that point B is achieved and

$$R_1 + R_2 < \log\left(1 + \frac{P_1 + P_2}{\sigma_0^2}\right) \quad (20)$$

Successive Interference Cancellation (SIC) Rx

Surprising observation: user 2 communicates at its individual capacity C_2 as if there were no user 1,

$$R_2 < \log\left(1 + \frac{P_2}{\sigma_0^2}\right) = C_2$$

while user 1 communicates at non-zero rate,

$$R_1 < \log\left(1 + \frac{P_1}{P_2 + \sigma_0^2}\right) < C_1$$

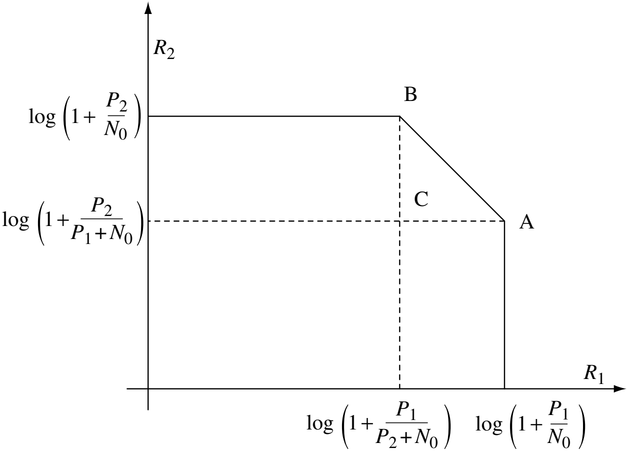
Q1: How to achieve point A?
Q2: Any point on line segment $[A, B]$?
Q3: Any point in the capacity region $\mathcal{C}$?

2-User MAC: Capacity Region

Q1: How to achieve point A?

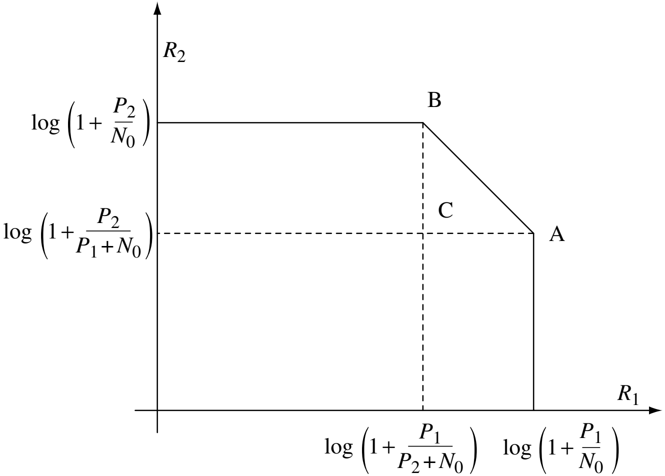
Q2: Any point on line segment $[A, B]$?

Q3: Any point in the capacity region $\mathcal{C}$?



2-User MAC: Capacity Region

Segment $[A, B]$ = Pareto optimal



SIC Rx is Optimal for MAC

Any point in $\mathcal{C}$ is achieved by the SIC Rx $=$ it is optimal.

K-User MAC

Channel model: SISO channel for each user,

$$y(t) = \sum_{k=1}^{K} x_k(t) + \xi(t) \tag{21}$$

Capacity region $\mathcal{C}$: a set of all rates $(R_1, .., R_K)$ satisfying

$$\sum_{k \in S} R_k < \log\left(1 + \frac{\sum_{k \in S} P_k}{\sigma_0^2}\right) \text{ for all } S \in \{1, .., K\} \tag{22}$$

S = any set of users.

Q1: work out the details for $K = 3$. Geometrically, what is S?

Q2: how to achieve all points in S?

2-User MAC: SIMO (Rx/BS Antenna Array)

The channel model is

$$\mathbf{y} = \mathbf{h}_1 x_1 + \mathbf{h}_2 x_2 + \boldsymbol{\xi} \tag{23}$$

The capacity region is

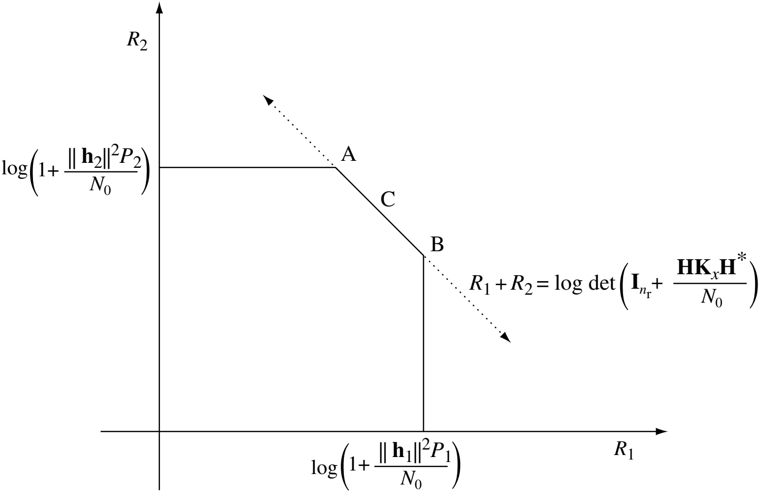
$$R_1 < \log\left(1 + \frac{|\mathbf{h}_1|^2 P_1}{\sigma_0^2}\right) = C_1 \tag{24}$$

$$R_2 < \log\left(1 + \frac{|\mathbf{h}_2|^2 P_2}{\sigma_0^2}\right) = C_2 \tag{25}$$

$$R_1 + R_2 < \log\left|\mathbf{I} + \sigma_0^{-2}\mathbf{W}\mathbf{P}\right| \tag{26}$$

where $\mathbf{W} = \mathbf{H}^+\mathbf{H}$, $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2]$, $\mathbf{P} = \text{diag}[P_1, P_2]$.

2-User SIMO MAC: Capacity Region[2]



[2] D. Tse, P. Viswanath, Fundamentals of Wireless Communications, Cambridge University Press, 2005; Ch. 6.1, 10.1

2-User SIMO MAC: Capacity Region

Q1: How to achieve A? B? [A,B]? Any point in $\mathcal{C}$?
Q2: How to recover individual user data?
Q3: Optimal Rx?

Successive Interference Cancellation (SIC) Rx

MAC Rx: How to recover individual (per-user) data?

Successive Interference Cancellation (SIC) Rx

MAC Rx: How to recover individual (per-user) data?

1. Detect user 1 treating user 2 as interference:

$$\mathbf{y} = \mathbf{h}_1 x_1 + \mathbf{h}_2 x_2 + \boldsymbol{\xi} \rightarrow y_{r1} = \mathbf{w}_1^+ \mathbf{y} = y_{s1} + y_{n1} \tag{27}$$

$y_{s1} = \mathbf{w}_1^+ \mathbf{h}_1 x_1 =$ signal part,
$y_{n1} = \mathbf{w}_1^+ (\mathbf{h}_2 x_2 + \boldsymbol{\xi}) =$ noise part,
$\mathbf{w}_1 = \mathbf{R}_{n1}^{-1} \mathbf{h}_1 = (\sigma_0^2 \mathbf{I} + P_2 \mathbf{h}_2 \mathbf{h}_2^+)^{-1} \mathbf{h}_1 =$ MMSE filter,

$$R_1 < \log(1 + \gamma_1) = C_1, \tag{28}$$

$$\gamma_1 = \frac{\overline{|y_{s1}|^2}}{\overline{|y_{n1}|^2}} = P_1 \mathbf{h}_1^+ (\sigma_0^2 \mathbf{I} + P_2 \mathbf{h}_2 \mathbf{h}_2^+)^{-1} \mathbf{h}_1 = \text{output filter SNR}$$

Successive Interference Cancellation (SIC) Rx

2. Subtract detected signal:

$$\mathbf{y}' = \mathbf{y} - \mathbf{h}_1 x_1 = \mathbf{h}_2 x_2 + \boldsymbol{\xi} \tag{29}$$

3. Detect user 2:

$$R_2 < \log\left(1 + \frac{P_2|\mathbf{h}_2|^2}{\sigma_0^2}\right) = C_2 \tag{30}$$

Successive Interference Cancellation (SIC) Rx

Q1: Show that

$$C_1 + C_2 = \log \left| \mathbf{I} + \sigma_0^{-2} \mathbf{W}\mathbf{P} \right| \tag{31}$$

i.e. point A is achieved.

Q2: How to achieve point B? [A,B]? Any point in $\mathcal{C}$?

K-User MAC

Channel model: SIMO channel for each user,

$$y(t) = \sum_{k=1}^{K} \mathbf{h}_k x_k(t) + \xi(t) \tag{32}$$

Capacity region $\mathcal{C}$: a set of all rates $(R_1, .., R_K)$ satisfying

$$\sum_{k\in S} R_k < \log \left| \mathbf{I} + \frac{1}{\sigma_0^2} \sum_{k\in S} P_k \mathbf{h}_k \mathbf{h}_k^+ \right| \text{ for all } S \in \{1, .., K\} \tag{33}$$

S = any set of users.

Q1: work out the details for $K = 3$. Geometrically, what is S?

Q2: how to achieve all points in S?

K-User MAC

The sum capacity:

$$\sum_{k=1}^{K} R_k < C_{sum} = \log \left| \mathbf{I} + \frac{1}{\sigma_0^2} \sum_{k=1}^{K} P_k \mathbf{h}_k \mathbf{h}_k^+ \right| = \log \left| \mathbf{I} + \sigma_0^{-2} \mathbf{WP} \right| \quad (34)$$

Q3: how does it compare to the MIMO channel capacity? Can the two be equal? If so, when?

Q4: assuming all per-user channels are orthogonal to each other, i.e. $\mathbf{h}_i^+ \mathbf{h}_j = 0$ for all $i \neq j$, find the capacity region explicitly. Explain how to achieve each point, and what precisely the SIC Rx is doing in this case.

Summary

- Multi-user systems: SDMA vs. FDMA
- Multiple access channel (MAC)
- Capacity region, sum and symmetric capacities
- SIMO MAC
- Successive interference cancellation (SIC) Rx

Reading

- D. Tse, P. Viswanath, Fundamentals of Wireless Communications: Ch. 6.1, 10.1.
- T.M. Cover, J.A. Thomas, Elements of Information Theory, Wiley, 2006 (2nd Ed.): Ch. 15.1-3 (14.1-3 in 1st Ed.).